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GODUNOV METHODS ON A STAGGERED MESH - AN IMPROVED ARTIFICIAL VISCOSITY

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ABSTRACT

We present a method for obtaining a High-Order, Non-Oscillatory hydro scheme for staggered mesh, internal energy-based codes using a simple modification of the standard von-Neumann artificial viscosity method. This modification is derived by applying the Godunov technique directly to a staggered mesh and replacing the full Riemann solution by a particular approximate Riemann solver. It will be shown to deliver results much better than the standard scheme and comparable to other modern hydro methods.

INTRODUCTION

Shock Hydrodynamics plays a key role in many applications. While the von Neumann-Richtmyer Artificial Viscosity method used to treat shocks in the vast majority of our codes is very general, easy to implement, and computationally efficient, it has several weaknesses: 1) It typically produces overly thick or oscillatory shock profiles, 2) It is incapable of properly treating weak shocks, and 3) It displays Noh's Q Errors.[1]

During the last decade or so, a number of high order differencing schemes for the equations of hydrodynamics have been developed, such as the Flux Corrected Transport (FCT),[2] Total Variation Diminishing (TVD),[3] and High Order Godunov methods,[4] that: 1) Give nearly optimally thin shocks (1 - 3 zones), 2) Have greatly reduced oscillations behind the shocks, 3) Accurately treat weak shocks, and 4) Typically display significantly smaller values of the Noh Q errors than Artificial Viscosity based methods.

Unfortunately, these methods, as presented, are not directly applicable to the staggered mesh, internal energy-based codes most commonly used today to solve the problems of continuum mechanics.

In the remainder of this paper, we will derive a High-Order, Non-Oscillatory hydro scheme for staggered mesh, internal energy-based codes that is a simple modification of the standard von-Neumann artificial viscosity method. This modification is obtained by applying the Godunov technique directly to a staggered mesh and replacing the full Riemann solution by a particular approximate Riemann solver. It will be shown to deliver results much better than the standard scheme and comparable to other modern hydro methods.

THE GODUNOV METHOD ON A SPATIALLY STAGGERED MESH

The starting point for any non-oscillatory, high-order scheme is a non-oscillatory first-order scheme. In this section we will apply the Godunov method directly to a spatially (not temporally) staggered mesh to obtain such a scheme. We will restrict our attention to a Lagrangian formulation, since the extension to an Eulerian formulation via a Lagrange plus remap approach is straightforward. Our procedure for applying the Godunov method to the staggered mesh (see fig. 1 for notation) has the following steps:

1) Solve the Riemann problems at the edges of the momentum control volumes (cell centers), where there are discontinuities in velocity (but not density, energy, or pressure). The pressure, PR_k, from the solution to the Riemann problem at these points can then be used to update the velocities, $W_{k+1/2}$:

$$\Delta W_{k+1/2} = \delta t \times (PR_k - PR_{k+1}) / M_{k+1/2} \quad , \qquad \text{(EQ 1)}$$

where $\delta t \equiv$ the time step, and

$$M_{k+1/2} = 0.5\,(M_k + M_{k+1}). \qquad \text{(EQ 2)}$$

2) Solve the Riemann problems at the cell edges, where there are discontinuities in density, energy and pressure (but not velocity). We will use the pressure, $PR_{k+1/2}$, and the velocity, $WR_{k+1/2}$, from the solution to the Riemann problem at these points to update the internal energy:

To do this, however, we must first update the total energy, E_k, using the $PR_{k+1/2}$ and $WR_{k+1/2}$:

$$\Delta E_k = \delta t\,(PR_{k-1/2}WR_{k-1/2} - PR_{k+1/2}WR_{k+1/2}) \qquad \text{(EQ 3)}$$

Use eq. 3 to update the internal energy using the relation:

$$\Delta e_k = \Delta E_k - \Delta KE_k \quad \text{, where} \qquad \text{(EQ 4)}$$

$$KE_k = M_k\,[W^2_{k-1/2} + W^2_{k+1/2}]\,/\,4. \qquad \text{(EQ 5)}$$

Some straightforward algebraic manipulation (using eq. 1, 2, 3, and 5) of eq. 4 gives the following expression for $\frac{\Delta e_k}{\delta t}$:

$$\frac{\Delta e_k}{\delta t} = PR_k \times (W^{n+1/2}_{k-1/2} - W^{n+1/2}_{k+1/2}) + (\Delta EF_{k-1/2} - \Delta EF_{k+1/2}) \qquad \text{(EQ 6)}$$

$$\Delta EF_{k+1/2} = [PR_{k+1/2}WR_{k+1/2} - PB_{k+1/2}\, W^{n+1/2}_{k+1/2}], \qquad \text{(EQ 7)}$$

$$PB_{k+1/2} = (M_{k+1}PR_k + M_kPR_{k+1}) / (M_k + M_{k+1}), \qquad \text{(EQ 8)}$$

$$\text{and} \quad W^{n+1/2}_{k+1/2} \equiv W^{n}_{k+1/2} + \frac{\Delta W_{k+1/2}}{2} \qquad \text{(EQ 9)}$$

Note that the first term on the RHS of eq. 6 is the same expression used in our "standard" method with PR $\Rightarrow$ P+Q. The second term does not affect total energy conservation, but rather represents a flux of energy between zones. Our "standard" codes simply ignore this term.

We have now generated expressions for updating both the velocities (and hence densities and positions) and the internal energies on a spatially staggered mesh in terms of the solutions to a set of Riemann problems defined on the two sets of control volumes. Our next step is to replace the full solutions to these Riemann problems by a particular approximate solution based on that suggested by Harten, Lax, and van Leer (HLL).[5]

HLL APPROXIMATE RIEMANN SOLVER

Since much of the detailed information provided by a full solution of the Riemann problem is lost in the projection step of a Godunov method, it is possible to replace the full solution by an approximate solution with little effect on the answer. Harten, Lax, and van Leer (HLL)[5] proposed an approximate Riemann solver with one intermediate state that needs as input, in addition to the initial conditions, only the lower and upper bounds, a_L and a_R, respectively, for the smallest and largest signal velocities.

For the Lagrangian equations of motion, $u_t + f(u)_M = 0$, where

$$u = \begin{bmatrix} V \\ W \\ E \end{bmatrix}, \quad f(u) = \begin{bmatrix} W \\ P \\ W \times P \end{bmatrix}, \text{ and V is the relative volume,} \qquad \text{(EQ 10)}$$

the HLL approximate solution to the Riemann problem is given by:

$$w(x/t, u_L, u_R) = \begin{pmatrix} u_L \text{ for } (x/t < a_L) \\ u_{LR} \text{ for } (a_L < x/t < a_R) \\ u_R \text{ for } (a_R < x/t) \end{pmatrix}, \qquad \text{(EQ 11)}$$

where

$$u_{LR} = [(a_Ru_R - a_Lu_L) - (f(u_R) - f(u_L))] / (a_R - a_L). \qquad \text{(EQ 12)}$$

In addition, we have the following expression for the flux:

$$f_{LR} = [(a_R f_L - a_L f_R) + a_R a_L (u_R - u_L)] / (a_R - a_L). \qquad \text{(EQ 13)}$$

In order to use this approximation we must make a choice for a_L and a_R. For reasons which will become clear, we make the following choice:

$$a_R = D_R \times C \qquad \text{(EQ 14)}$$

$$a_L = -D_L \times C$$

The quantity C in eq. 14 is an estimate of the maximum speed of the signals generated in the interaction of the left and right hand sides (The density factors are present because we are using Lagrangian coordinates).

A useful approximation for this quantity in the vicinity of a shock is

$$C = C_s + \alpha |W_R - W_L| \quad , \qquad \text{(EQ 15)}$$

where C_S is the sound speed of the *unshocked* material and α is a material dependent quantity which can be derived from the equation of state or determined empirically from a particle-velocity/shock-velocity plot (A similar approximation was used by Dukowicz[6] in his derivation of an approximate Riemann solver).

Combining eq. 12-15 gives us the following expressions for P_{LR} and W_{LR}:

$$P_{LR} = \frac{D_R P_L + D_L P_R}{D_R + D_L} - \frac{D_L D_R}{D_R + D_L} (C_s \Delta W + \alpha |\Delta W| \Delta W) \qquad \text{(EQ 16)}$$

where $\Delta W = |W_R - W_L|$, and (EQ 17)

$$W_{LR} = \frac{D_R W_R + D_L W_L}{D_R + D_L} - \frac{(P_R - P_L)}{(C_s + \alpha |\Delta W|)\ (D_R + D_L)} \qquad \text{(EQ 18)}$$

Using eq. 16 and 18 we can finally generate expressions for the quantities we need in eq. 1 and 6:

$$PR_k = P_k - 0.5 D_k (C_{s,k} \Delta W_k + \alpha_k |\Delta W_k| \Delta W_k) \qquad \text{(EQ 19)}$$

$$PR_{k+1/2} = \frac{D_{k+1} P_k + D_k P_{k+1}}{D_{k+1} + D_k} \qquad \text{(EQ 20)}$$

$$WR_{k+1/2} = W_{k+1/2} - \frac{(P_R - P_L)}{MIN(C_{s,k}, C_{s,k+1})\,(D_k + D_{k+1})} \qquad \text{(EQ 21)}$$

THE CONNECTION WITH ARTIFICIAL VISCOSITY

We must make one more approximation to complete the connection with our "standard" artificial viscosity based codes. This approximation involves setting the energy flux terms in eq. 6 equal to zero. (Note that this does not destroy conservation of energy or alter the rate at which kinetic energy is transformed into internal energy.)

This approximation is equivalent to moving the zone boundaries with the velocity $W^{n+1/2}_{k+1/2}$ rather than with $WR_{k+1/2}$, and replacing $PR_{k+1/2}$ with $PB_{k+1/2}$. Observe that for uniform zoning, $PR_{k+1/2}$ and $PB_{k+1/2}$ are identical anyway.

Given this last approximation we see that we have arrived at the same set of equations used in a "standard", first-order, staggered-mesh Lagrangian scheme if we make the following substitution:

$$Q_k \Rightarrow -.5\,D_k\,(C_{s,k}\Delta W_k + \alpha_k|\Delta W_k|\Delta W_k) \qquad \text{(EQ 22)}$$

We have now arrived at a most interesting result:

Our "standard" staggered mesh artificial viscosity method can be made completely equivalent (modulo the energy flux terms) to a staggered mesh Godunov method based on the HLL approximate Riemann solver by simply choosing the coefficients of the linear and quadratic terms in accordance with eq. 22.

SECOND ORDER EXTENSIONS

For the sake of simplicity, we will not explicitly tackle the problem of staggered temporal meshes. In practice we have found that using the existing temporal centering in the Q formulations of temporally staggered mesh codes in conjunction with the modified spatial formulations described below has been quite satisfactory. Further work on this problem is underway.

Given the close connection we have established between our "standard" artificial viscosity and the Godunov method, it is now straightforward to use the same techniques that have been used to obtain spatially higher-order extensions of the Godunov method to do the same to our "standard" method. We have implemented schemes based both on the "pre-processing" methods of van Leer[4] and the "post-processing" TVD methods of Harten[3]. While each has its own strengths and weaknesses, they can all be made to work well.

One simple second-order extension to implement is based on a TVD approach. In this approach we simply express our fluxes (in this case P + Q), as the sum of a first-order part (Q) and a second-order part (P) and put a multiplier on the first-order part that goes to one in the neighborhood of rapid oscillations in the velocity field (this includes shocks) and to zero elsewhere.

While formally we can achieve second order accuracy by limiting only the linear term in the artificial viscosity, in practice we have found it beneficial to also limit the quadratic term, albeit more gently:

$$\begin{pmatrix} \Delta W \\ |\Delta W|\Delta W \end{pmatrix} \Rightarrow \begin{pmatrix} \Delta W \times (1-\phi) \text{ in the linear term, and} \\ |\Delta W|\Delta W \times (1-\phi^2) \text{ in the quadratic term} \end{pmatrix} \qquad \text{(EQ 23)}$$

where ϕ is the TVD limiter:

$$\phi_k = \max(0, \min(.5(R_L + R_R), 2R_L, 2R_R, 1)), \qquad \text{(EQ 24)}$$

$$R_L = \left(\frac{\partial W}{\partial x}_L\right) / \left(\frac{\partial W}{\partial x}_C\right) \quad , \quad R_R = \left(\frac{\partial W}{\partial x}_R\right) / \left(\frac{\partial W}{\partial x}_C\right) \quad , \qquad \text{(EQ 25)}$$

$$\left(\frac{\partial W}{\partial x}_L\right) = \frac{W_{k-1/2} - W_{k-3/2}}{x_{k-1/2} - x_{k-3/2}} , \quad \left(\frac{\partial W}{\partial x}_C\right) = \frac{W_{k+1/2} - W_{k-1/2}}{x_{k+1/2} - x_{k-1/2}} , \qquad \text{(EQ 26)}$$

$$\left(\frac{\partial W}{\partial x}_R\right) = \frac{W_{k+3/2} - W_{k+1/2}}{x_{k+3/2} - x_{k+1/2}} \quad . \qquad \text{(EQ 27)}$$

This completes the spatially second-order accurate extension of our new method. In practice, applying this method to existing staggered mesh, artificial viscosity based codes allows them to perform competitively with existing second-order Godunov based codes with considerably less computational effort and complexity. Shock widths are decreased by 30% to 50%, and post-shock oscillations are eliminated or significantly reduced. A number of previously mentioned classic difficulties, such as wall heating, radioed zone errors, and spurious convergence heating are similarly reduced.

EXTENSION TO MULTIPLE DIMENSIONS

Once again following in the path of existing high-order Godunov methods, we have used operator splitting as the main way of extending our new method to multiple dimensions.

This leads to a "split Q", where there is a separate Q (with a separate limiter) for each velocity component. For codes with face-centered velocities, the previous 1D derivation can be applied without modification. For codes with node-centered velocities the quantities D_k and $C_{s,k}$ in eq. 22 must be replaced by a simple average.

Although we have found this "split Q" approach to be quite satisfactory, we have developed an un-split, fully tensor-invariant formulation (but that's another paper).

CONCLUSION

We have demonstrated the close relationship between the Godunov method on a staggered mesh and our "standard" methods and used this connection, in conjunction with the HLL approximate Riemann solver, to develop a Non-Oscillatory, High Resolution hydro method using artificial viscosity

This method has been applied to existing 1D Lagrange codes, 2D face-centered and node-centered Eulerian codes and 2D node-centered Lagrange and ALE codes. The method has proven easy to apply and computationally economical to use.

The results obtained with these codes (see below) are equivalent in quality to those obtained by the better known cell-centered, total-energy based FCT, high-order Godunov, and TVD codes.

ACKNOWLEDGMENTS

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Fig. 1. Variable Placement on Staggered Spatial Grid

Fig. 2. Density Curves for New Q and von Neumann Q on Noh Problem (same zoning)

Fig. 3. Density Curves for New Q (doubled zone size) and the von Neumann Q.

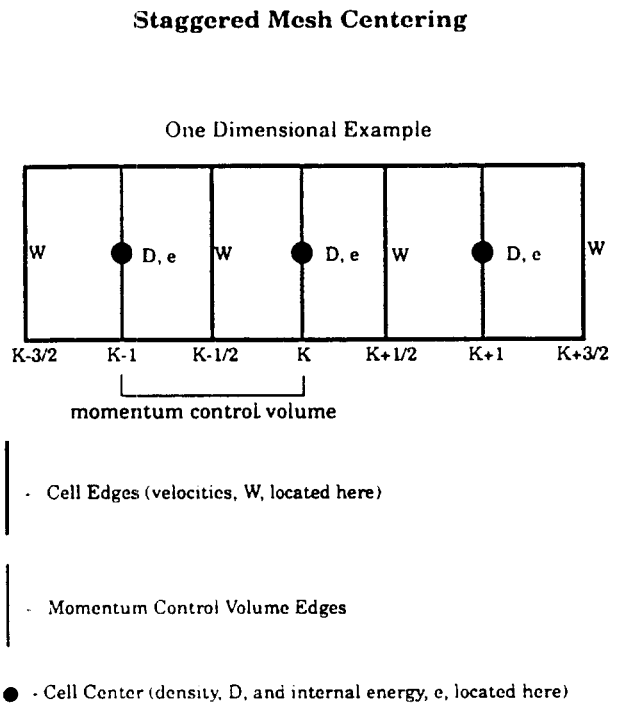
Fig. 4. Energy Curves for New Q and von Neumann Q (same zoning)

Fig. 5. One-Dimensional Blast Wave Problem of Woodward and Colella - Initial Condition

Fig. 6. Velocity Plot of Blast Wave Prob. at t = .018. Note only two cells across shock.

Fig. 7. Density Plot of Blast Wave Problem at t = .036 for von Neumann Q, New Q First Order, and New Q Second Order.

Fig. 8. Density Plot for Andronov Shock Tube Problem (weak shock).



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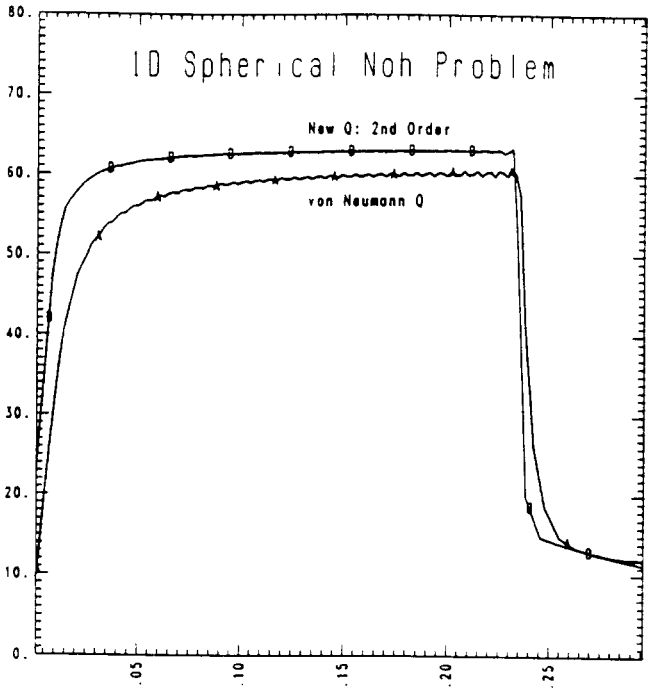

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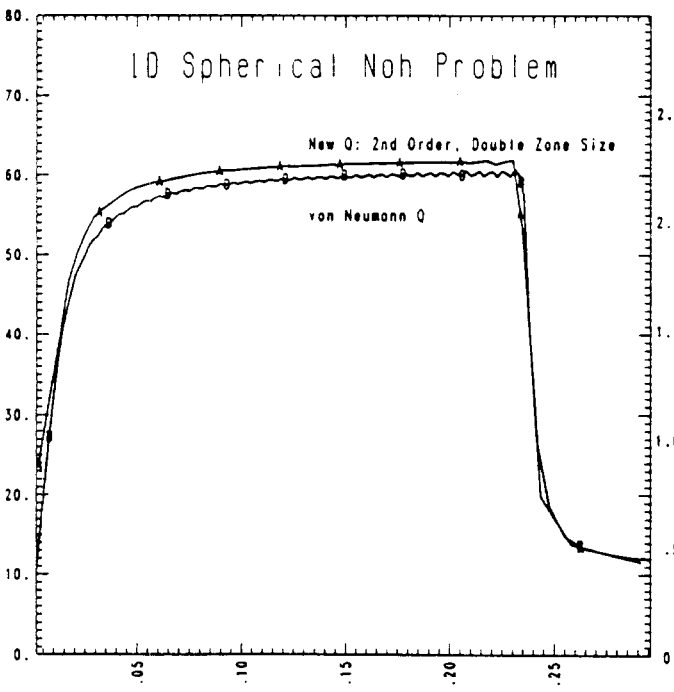

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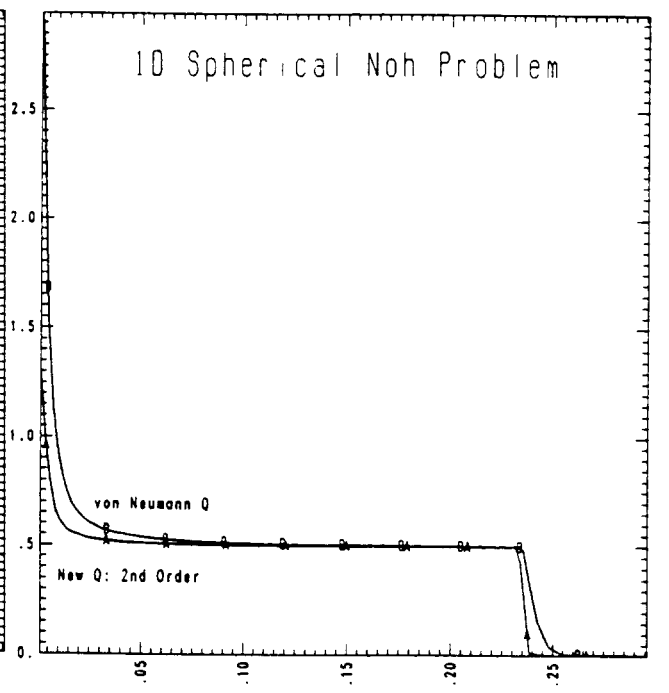

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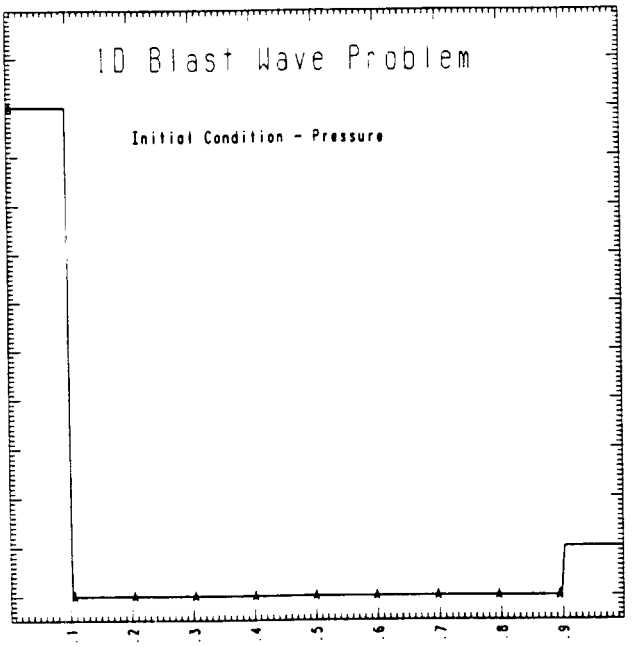

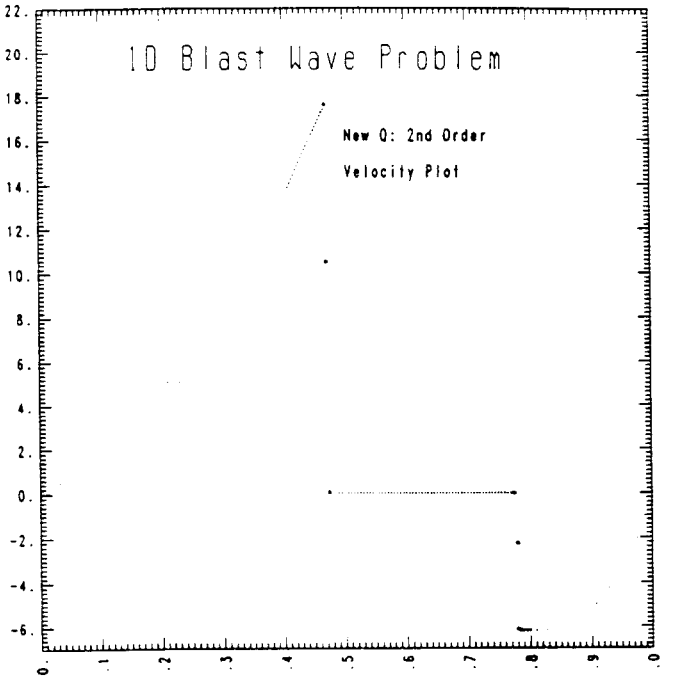

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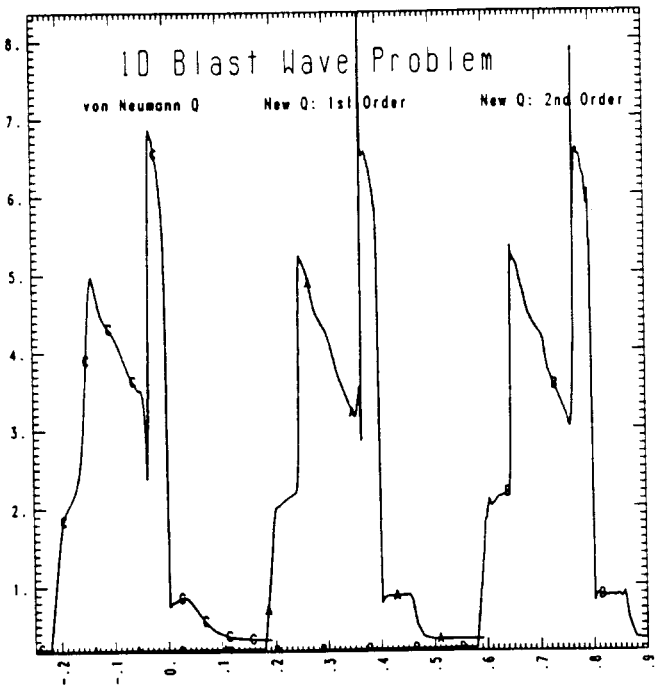

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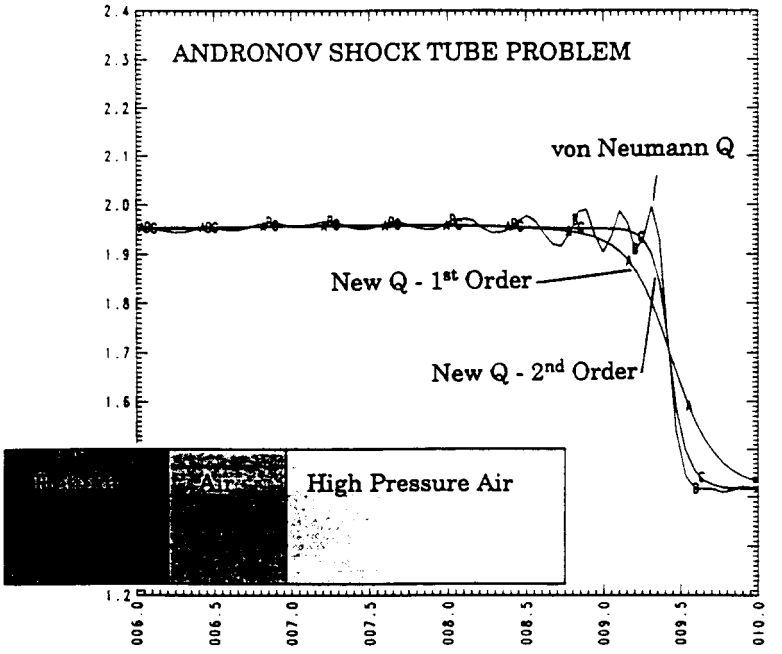

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