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Coupling mixed and finite volume discretizations of convection-diffusion-reaction equations on non-matching grids

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ABSTRACT *In this paper, we consider approximation of a second order convection-diffusion problem by coupled mixed and finite volume methods. Namely, the domain is partitioned into two subdomains, and in one of them we apply the mixed finite element method while on the other subdomain we use the finite volume element approximation. We prove the stability of this discretization and derive an error estimate.*

Key Words: combined mixed and finite volume methods, non–matching grids.

1. Introduction

Coupling different numerical methods applied to different parts of the domain of interest is becoming an important tool in numerical analysis, scientific computing, and engineering simulations. In the coupling process several important mathematical issues arise that have to be addressed.

First problem is to find what natural and stable mathematical formulation will lead to a good computational scheme. In the case of different methods used in different parts of the domain this means to find a stable way of gluing together the solutions in the subdomains. Secondly, we have to find an approximation of the mathematical formulation which is stable, convergent, and accurate. And finally, we have to construct and study efficient solution methods for the resulting algebraic problem.

We shall consider the following homogeneous Dirichlet boundary value problem for the convection-diffusion-reaction equation:

$$\mathcal{L}p = f(x), \quad x \in \Omega, \qquad p(x) = 0, \quad x \in \partial\Omega, \tag{1}$$

where $\mathcal{L} = \mathcal{L}_0 + \mathcal{C}$, $\mathcal{L}_0 p \equiv -\nabla \cdot a\nabla p$ is the diffusion operator, and $\mathcal{C}p \equiv \nabla \cdot (p\underline{b}) + c_0 p$ is the convection-reaction operator. Here Ω is a bounded polygon in $\mathcal{R}^d$, $d = 2, 3$ with a boundary $\partial\Omega$, $a = a(x) = \{a_{i,j}(x)\}$ is a $d \times d$ symmetric and uniformly in Ω positive definite matrix, and $f = f(x)$ is a known function in $L^2(\Omega)$. Also $\underline{b} = \underline{b}(x) = (b_1, \cdots, b_d)$ is a given vector field and $c_0 = c_0(x)$ is a given function. We assume that $\underline{b}(x)$ and $c_0(x)$ are uniformly bounded in Ω and satisfy the condition

$$c_0(x) + \frac{1}{2}\nabla \cdot \underline{b}(x) \geq \gamma_0 = \text{const} > 0, \quad x \in \Omega. \tag{2}$$

This in turn guarantees the coercivity of the operator $\mathcal{C}$ in $L^2(\Omega)$ and the existence and uniqueness of its solution in the Sobolev space $H_0^1(\Omega)$. This problem is a prototype of mathematical models in heat and mass transfer, diffusion-reaction processes, flow and transport in porous media, etc.

In this paper we propose and study numerical methods for this problem when in different parts of the domain different discretizations on independent meshes are used. Namely, we consider mixed finite element approximation in one part of the domain and finite volume element method in the rest of the domain. It is important to note that coupling mixed finite element and finite volume or Galerkin finite element approximations does not require any auxiliary (mortar) space on the interface of the subdomains. This is due to the fact that the Dirichlet boundary conditions are natural for the mixed formulation, while the Neumann boundary conditions are natural for the standard weak formulation of a second order elliptic problem.

In the recent years there has been growing interest in the finite volume method (called also control-volume method or box-schemes). This interest is mostly due to the requirement of many applications of having locally conservative discretizations. This is a discrete variant of the property of the continuous model which expresses conservation of certain quantity (mass, heat, momentum, etc) over each infinitesimal volume. The finite volume method has been combined with the technique of the finite element method in a new development which is capable of producing accurate approximations on general simplicial and quadrilateral grids (see, e.g. [4, 5, 6, 7, 8, 13]). For a collection of theoretical results and various applications we refer [2]. The main advantages of the finite volume method are compactness of the discretization stencil, good accuracy, and discrete local conservation, which for many applications is a very desirable feature of the approximation. Also, this method has well developed approximation schemes for convection and convection-dominated problems.

The structure of the paper is the follow. In Section 2 we introduce all necessary notations and the weak form of the problem (1) in a two domains setting. In the subdomain where the mixed formulation is used we apply the more general concept of the discontinuous Galerkin method for second order equations in mixed form. In the case of convection-diffusion problems the pressure should be smoother than just L^2 so we use the space H^1_{loc}. Further, we study the stability and derive an a priori estimate for the solution.

Section 3 is the central part of the paper. Here we introduce and study the coupling of the mixed finite element and the finite volume element method. Further, in Subsection 3.2 we discuss the coupled mixed and finite volume approximation of convection-diffusion-reaction equations. Finally, in Subsection 3.3 we prove the unconditional stability of the discrete scheme and derive an estimate for the error.

2. Variational formulation

In this section we first introduce all necessary notations for splitting the domain of the problem (1) in two subdomains $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and using two different formulations in each subdomain. The weak mixed formulation in Ω_1 is derived when the pressure p is in the space $H^1_{loc}(\Omega_1)$. In Ω_2 we use pressure space $H^1(\Omega_2)$. We prove that the coupled mixed/Galerkin formulation is stable and derive an a priori estimate which is the prototype of estimates for the approximations schemes established further in the paper.

2.1. Two-subdomain coupled formulation

We partition Ω into two subdomains with an interface boundary Γ, i.e. $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ (see Figure 1) and use the standard notations for Sobolev spaces of functions defined on Ω_1 and Ω_2: $H(\text{div},\ \Omega_1)$, $L^2(\Omega_i)$, $i = 1, 2$ and $H^1_0(\Omega_2,\ \partial\Omega_2 \setminus \Gamma)$. Here the last space denotes the functions defined on Ω_2 having generalized derivatives in $L^2(\Omega_2)$ and vanishing on $\partial\Omega_2 \setminus \Gamma$. The inner products in these spaces are denoted correspondingly by

$$(\mathbf{u}, \mathbf{v})_{H(\text{div},\, \Omega_1)} \equiv \int_{\Omega_1} (\mathbf{u} \cdot \mathbf{v} + \nabla \mathbf{u}\ \nabla \mathbf{v}) dx, \qquad (w_i, z_i) = \int_{\Omega_i} w_i, z_i dx,$$

and $(v_2, w_2)_{H^1(\Omega_2)} = (v_2, w_2) + (\nabla v_2, \nabla w_2)$, which in turn define norms denoted by $\|\mathbf{v}\|_{H(\text{div},\, \Omega_1)}$, $\|v_i\|_{0,\, \Omega_i}$ and $\|v_2\|_{1,\, \Omega_2}$.

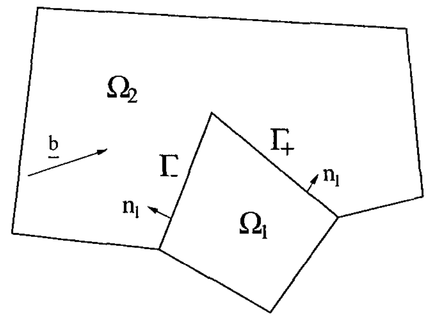

Figure 1: Domain partitioning: $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$.

Whenever possible we skip the subscript in L^2-norms. The dual space of $H_0^1(\Omega_2,\ \partial\Omega_2 \setminus \Gamma)$ is denoted by $H^{-1}(\Omega_2)$, the space of the traces of the normal component of the vector-functions in $H(\mathrm{div},\ \Omega_1)$ is denoted by $H^{-1/2}(\Gamma)$, and the space of the traces of functions in $H_0^1(\Omega_2,\ \partial\Omega_2 \setminus \Gamma)$ is denoted by $H_{00}^{1/2}(\Gamma)$. The trace spaces are equipped with the standard Sobolev norms. Finally, we shall use the notation $< \cdot, \cdot >_\Gamma$ for the duality pairing between $H_{00}^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. Further, we denote by $\mathbf{n}_i$ the unit vector normal to $\partial\Omega_i$ for $i = 1, 2$ and pointing outward to the domain. Finally, we split the interface boundary $\Gamma = \Gamma_- \cup \Gamma_+$ where $\Gamma_- = \{x \in \Gamma :\ \underline{b}(x) \cdot \mathbf{n}_1 < 0\}$ and $\Gamma_+ = \{x \in \Gamma :\ \underline{b}(x) \cdot \mathbf{n}_1 \geq 0\}$. Note, that this splitting is with respect to the vector $\mathbf{n}_1$. An illustration of these notations in 2-D is given on Figure 1.

In Ω_1 we use a mixed setting of the problem (1). That is, we introduce the new (vector) variable $\mathbf{u} = -a\nabla p$. To distinguish the solutions in the subdomains we denote by $p_1 = p|_{\Omega_1}$ and $p_2 = p|_{\Omega_2}$. The composite model will impose different smoothness requirements on the components p_1 and p_2. More specifically, we will require that $\mathbf{u} \in H(\mathrm{div},\ \Omega_1)$, $p_1 \in L^2(\Omega_1)$, and $p_2 \in H_0^1(\Omega_2,\ \partial\Omega \setminus \Gamma)$. Note that p_2 is required to vanish on $\partial\Omega_2 \setminus \Gamma$.

Testing the equation $a^{-1}\mathbf{u} + \nabla p_1 = 0$ by a function $\mathbf{v} \in H(\mathrm{div},\ \Omega)$, using integration by parts, the zero boundary conditions for p_1 on $\partial\Omega_1 \setminus \Gamma$, and the fact the trace of p_1 on Γ is the same for the trace of p_2 on Γ, one ends up with the equation,

$$(a^{-1}\mathbf{u},\ \mathbf{v}) - (p_1, \nabla \cdot \mathbf{v}) + < p_2,\ \mathbf{v} \cdot \mathbf{n}_1 >_\Gamma = 0 \quad \text{for all } \mathbf{v} \in H(\mathrm{div},\ \Omega_1). \tag{3}$$

Further, in order to describe the weak form of the equation

$$\nabla \cdot \mathbf{u} + \nabla \cdot (\underline{b} p_1) + c_0 p_1 = f(x) \quad \text{in } \Omega_1$$

we need to allow discontinuous functions p_1 from the space H^1_{loc}:

$$H^1_{loc}(\Omega_1) = \left\{ \begin{array}{ll} v_1 \in L^2(\Omega_1): & \text{there is a partition } \mathcal{K} \text{ of } \Omega_1 \\ & \text{such that } v_1|_K \in H^1(K) \text{ for all } K \in \mathcal{K} \end{array} \right\}.$$

The functions in $H^1_{loc}(\Omega_1)$ have traces from both sides of the interfaces of the subdomains K. Namely, for a given function $p_1 \in H^1_{loc}(\Omega_1)$ we denote these traces by p_1^o and p_1^i, where "o" stands for the outward (with respect to K) trace and respectively, "i" stands for the interior trace.

Next, we give the weak form of the above equation. We borrow this formulation from the discontinuous Galerkin methods (see, e.g. [10], pp. 189-196) by testing the equation by a function $w_1 \in H^1_{loc}(\Omega_1)$. We note, that this setting is quite similar to the mixed finite element method for convection-dominated convection-diffusion-reaction equations (see, e.g. [12]). Since the functions from $H^1_{loc}(\Omega_1)$ are piece-wise smooth with respect to the partition $\mathcal{K}$ we shall integrate over each $K \in \mathcal{K}$ and then sum the results. Following [10] we find first the contributions of the advection-reaction operator $\mathcal{C}p_1$ by introducing the bilinear form $C_K(p_1,\, w_1)$ for any subdomain $K \in \mathcal{K}$:

$$C_K(p_1,\, w_1) = \int_K \left(\nabla \cdot (\underline{b} p_1) + c_0(x) p_1\right) w_1 \; dx + \int_{\partial K_-} (p_1^o - p_1^i)\; w_1^i \; \underline{b} \cdot \mathbf{n} \; ds.$$

Here $\mathbf{n}$ is the outer unit normal vector to ∂K. Next, we integrate by parts in each subdomain K and sum over all $K \in \mathcal{K}$. Thus, for $p_1, w_1 \in H^1_{loc}(\Omega_1)$ we get:

$$\begin{aligned} C(p_1, w_1) = \sum_{K \in \mathcal{K}} \Big(& -\int_K p_1 \; \underline{b} \cdot \nabla w_1 \; dx + \int_{\partial K_-} p_1^o \; w_1^i \; \underline{b} \cdot \mathbf{n} \; ds \\ & + \int_K c_0(x) \; p_1 \; w_1 dx + \int_{\partial K_+} p_1^i \; w_1^i \; \underline{b} \cdot \mathbf{n} \; ds \Big). \end{aligned} \tag{4}$$

Note that this bilinear from is well defined for both continuous and discontinuous functions with respect to the partition $\mathcal{K}$. From this expression we see that if the subdomain K has a side/face on Γ_- then the trace p_1^o should be replaced by its counterpart from Ω_2, namely by $p_2(x)$. Also on Γ_- we have $w_1^i = w_1$ and on $\partial\Omega_{1-} \setminus \Gamma_-$ we take $p_1^o = 0$. Further, for a given function $t(x)$ we denote by $t_- = \min(0, t)$ and $t_+ = \max(0, t)$. Thus, we get the following weak form of the second equation valid for all $w_1 \in H^1_{loc}(\Omega_1)$:

$$-(\nabla \cdot \mathbf{u},\, w_1) - a_{11}(p_1,\, w_1) - a_{12}(p_2,\, w_1) = -(f, w_1), \tag{5}$$

where

$$a_{11}(p_1,\ w_1) = \sum_{K\in\mathcal{K}} \int_{\partial K\setminus\partial\Omega_{1-}} \left((\underline{b}\cdot\mathbf{n})_-\ p_1^o + (\underline{b}\cdot\mathbf{n})_+\ p_1^i\right) w_1^i\ ds$$
$$- \sum_{K\in\mathcal{K}} \int_K p_1\ \underline{b}\cdot\nabla w_1\ dx + (c_0 p_1,\ w_1) \text{ for } p_1,\ w_1 \in H^1_{loc}(\Omega_1) \tag{6}$$

and

$$a_{12}(p_2,\ w_1) = \int_{\Gamma_-} p_2(x)\ w_1(x)\ \underline{b}\cdot\mathbf{n}_1\ ds \text{ for } v_2 \in W_2,\ w_1 \in H^1_{loc}(\Omega_1). \tag{7}$$

Finally, testing the equation (1) by a function $w_2 \in H^1_0(\Omega_2;\ \partial\Omega_2\setminus\Gamma)$, using integration by parts, the zero boundary condition for w_2 on $\partial\Omega_2\setminus\Gamma$, and the fact that $\mathbf{u}\cdot\mathbf{n}_1 = -a\nabla p_1\cdot\mathbf{n}_1 = a\nabla p_2\cdot\mathbf{n}_2$ on Γ, one arrives at,

$$< \mathbf{u}\cdot\mathbf{n}_1,\ w_2 >_\Gamma - (a\nabla p_2,\ \nabla w_2) - (\nabla\cdot(\underline{b}p_2), w_2) - (c_0 p_2, w_2) = -(f,\ w_2), \tag{8}$$

for all $w_2 \in H^1_0(\Omega_2;\ \partial\Omega_2\setminus\Gamma)$.

There are various ways one can take into account the influence of the problem in the domain Ω_1 on the problem in Ω_2. One of the possibilities, which we shall use further, is to try to make a formulation, which is stable for small diffusion coefficient (or even for vanishing diffusion). In this case it is very important to formulate correctly the boundary conditions. Namely, at the "inflow" part of the interior boundary the solution should be specified from the "outside" data. Taking into account that Γ_+ is the "inflow" part of Γ for the subdomain Ω_2, we add $\int_{\Gamma_+} p_1 w_2\ \underline{b}\cdot\mathbf{n}_2\ ds$ and subtract its equal $\int_{\Gamma_+} p_2 w_2\ \underline{b}\cdot\mathbf{n}_2\ ds$ since on Γ we have $p_1 = p_2$. Thus, we get the following form of the last equation:

$$< \mathbf{u}\cdot\mathbf{n}_1,\ w_2 >_\Gamma - a_{21}(p_1, w_2) - a_{22}(p_2, w_2) = -(f, w_2), \tag{9}$$

for all $w_2 \in H^1_0(\Omega_2;\ \partial\Omega_2\setminus\Gamma)$, where

$$a_{21}(p_1, w_2) = \int_{\Gamma_+} p_1 w_2\ \underline{b}\cdot\mathbf{n}_2\ ds, \tag{10}$$

$$a_{22}(p_2, w_2) = (a\nabla p_2,\ \nabla w_2) + (\nabla\cdot(\underline{b}p_2), w_2) + (c_0 p_2, w_2) - \int_{\Gamma_+} p_2 w_2\ \underline{b}\cdot\mathbf{n}_2\ ds. \tag{11}$$

Thus, the coupled system for the three unknowns $\mathbf{u} \in H(\text{div},\ \Omega_1)$, $p_1 \in H^1_{loc}(\Omega_1)$ and $p_2 \in H^1_0(\Omega_2;\ \partial\Omega_2\setminus\Gamma)$ consists of the equations (3), (5), and (9) summarized as:

$$\left\{\begin{array}{llll} (a^{-1}\mathbf{u},\ \mathbf{v}) & -(p_1, \nabla\cdot\mathbf{v}) & + < p_2,\ \mathbf{v}\cdot\mathbf{n}_1 >_\Gamma & = 0, \\ -(\nabla\cdot\mathbf{u},\ w_1) & -a_{11}(p_1, w_1) & -a_{12}(p_2, w_1) & = -(f, w_1), \\ < \mathbf{u}\cdot\mathbf{n}_1,\ w_2 >_\Gamma & -a_{21}(p_1, w_2) & -a_{22}(p_2, w_2) & = -(f,\ w_2), \end{array}\right. \tag{12}$$

for all $\mathbf{v} \in H(\mathrm{div},\ \Omega_1)$, $w_1 \in H^1_{loc}(\Omega_1)$, and $w_2 \in H^1_0(\Omega_2;\ \partial\Omega_2\setminus\Gamma)$, respectively. The bilinear forms $a_{ij}(\cdot,\cdot)$ are defined by (6), (7), (10), and (11), respectively.

2.2. Well-posed-ness of the composite problem

Here we verify the existence and uniqueness of the solution of problem (12) and its stability in an appropriate norm. For this we shall need some additional notations. Let $\mathcal{E} = \{e\}$ be the set of edges/faces of the subdomain Ω_1 from $\mathcal{K}$ and $\mathcal{E}_0$ the set of interior for Ω_1 edges/faces. Recall, that $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outward unit normal vectors to Ω_1 and Ω_2, respectively. For any edge $e \in \mathcal{E}_0$ denote by $\mathbf{n}_e$ a fixed unit vector normal to e and let K_e^+ and K_e^- be the two adjacent to e subdomains from the partition $\mathcal{K}$. For edges/faces that are on $\partial\Omega_1$ we shall always assume that $\mathbf{n}_e = \mathbf{n}_1$. Further, denote by $[v_1]$ and $\overline{v}_1$ the jump and the average of the discontinuity of v_1, respectively, along any edge e. More precisely, this is the difference and the arithmetic mean of the traces $v_1|_{K_e^+}$ and $v_1|_{K_e^-}$ taken from both sides of e:

$$[v_1] = v_1|_{K_e^+} - v_1|_{K_e^-}, \qquad \overline{v}_1 = \frac{1}{2}(v_1|_{K_e^+} + v_1|_{K_e^-}).$$

Further, we use the following natural norm for $v_1 \in H^1_{loc}(\Omega_1)$ and $v_2 \in H^1_0(\Omega_2;\ \partial\Omega_2 \setminus \Gamma)$:

$$\begin{aligned} \|v_1\|^2_{*,\Omega_1} + \|v_2\|^2_{*,\Omega_2} &= \frac{1}{2}\sum_{e\in\mathcal{E}_0}\int_e [v_1]^2|\underline{b}\cdot\mathbf{n}|\,ds + \gamma_0(\|v_1\|^2_{0,\Omega_1} + \|v_2\|^2_{0,\Omega_2}) \\ &+ \frac{1}{2}\int_{\partial\Omega_{1+}\setminus\Gamma_+} v_1^2\,\underline{b}\cdot\mathbf{n}_1\,ds - \frac{1}{2}\int_{\partial\Omega_{1-}\setminus\Gamma_-} v_1^2\,\underline{b}\cdot\mathbf{n}_1\,ds \\ &+ \frac{1}{2}\int_{\Gamma_+}(v_1 - v_2)^2\underline{b}\cdot\mathbf{n}_1\,ds - \frac{1}{2}\int_{\Gamma_-}(v_1 - v_2)^2\underline{b}\cdot\mathbf{n}_1\,ds \\ &+ (a\nabla v_2, \nabla v_2). \end{aligned} \tag{13}$$

All terms in the expression on the right are nonnegative and this defines a norm on the space $H^1_{loc}(\Omega_1) \times H^1_0(\Omega_2;\ \partial\Omega_2 \setminus \Gamma)$. Note, that under certain conditions on the vector field $\underline{b}$ this is a norm even if $\gamma_0 = 0$.

The stability of the composed problem (12) is based on the following theorem:

Theorem 1 *The solution of the problem (12) satisfies the a priori estimate:*

$$\|\mathbf{u}\|^2_{L^2(\Omega_1)} + \|p_1\|^2_{*,\,\Omega_1} + \|p_2\|^2_{*,\,\Omega_2} \le C\|f\|^2_{0,\,\Omega}, \tag{14}$$

where the $*$-norm is defined by (13).

The proof of the above theorem is based on the following lemmas.

Lemma 1 *The bilinear form (6) defined for $v_1,\ w_1 \in H^1_{loc}(\Omega_1)$ can be transformed to the following form:*

$$\begin{aligned} a_{11}(v_1, w_1) &= \frac{1}{2}\sum_{e\in\mathcal{E}_0}\int_e [v_1][w_1]|\underline{b}\cdot\mathbf{n}|\,ds + \int_{\partial\Omega_{1+}} \underline{b}\cdot\mathbf{n}_1\, v_1\, w_1\, ds + (c_0 v_1,\ w_1) \\ &+\frac{1}{2}\sum_{e\in\mathcal{E}_0}\int_e \underline{b}\cdot\mathbf{n}_e[v_1](w_1|_{K_e^+} + w_1|_{K_e^-})\,ds - \sum_{K\in\mathcal{K}}\int_K v_1\,\underline{b}\cdot\nabla w_1 dx. \end{aligned}$$

Furthermore,

$$\begin{aligned} a_{11}(v_1, v_1) = {} & \frac{1}{2}\sum_{e\in\mathcal{E}_0}\int_e [v_1]^2|\underline{b}\cdot\mathbf{n}_e|\,ds + ((c_0 + \frac{1}{2}\nabla\cdot\underline{b})v_1,\ v_1) \\ & +\frac{1}{2}\int_{\partial\Omega_{1+}} v_1^2\,\underline{b}\cdot\mathbf{n}_1\,ds - \frac{1}{2}\int_{\partial\Omega_{1-}} v_1^2\,\underline{b}\cdot\mathbf{n}_1\,ds. \end{aligned}$$

Lemma 2 *For all $v_2 \in H^1_0(\Omega_2,\ \partial\Omega_2\setminus\Gamma)$*

$$\begin{aligned} a_{22}(v_2, v_2) = (a\nabla v_2,\ \nabla v_2) \quad &+ \quad ((c_0 + \tfrac{1}{2}\nabla\cdot\underline{b})v_2, v_2) \\ &+ \quad \frac{1}{2}\int_{\Gamma_+} v_2^2\,\underline{b}\cdot\mathbf{n}_1\,ds - \frac{1}{2}\int_{\Gamma_-} v_2^2\,\underline{b}\cdot\mathbf{n}_1\,ds. \end{aligned}$$

3. Coupling mixed and finite volume approximations of the convection-diffusion equation

Our approximation strategy is based on the finite volume method in the framework studied by Cai [4], Cai, Mandel, and McCormick [6] and also by Bank and Rose [1].

3.1. An outline of the finite volume element method

We first outline a finite volume discretization method for the case of pure diffusion problem posed on Ω_2,

$$-\nabla\cdot a\nabla p_2 = f(x) \quad \text{in } \Omega_2, \;\; p_2 = 0 \quad \text{on } \partial\Omega_2\backslash\Gamma, \;\; -a\nabla p_2\cdot\mathbf{n}_2 = \eta_N \quad \text{on } \Gamma. \tag{15}$$

Here $f \in L^2(\Omega_2)$ is given and η_N, for the time being is assumed given in the space $L^2(\Gamma)$. The finite volume method under consideration uses two different finite dimensional spaces: a solution space W_2 and the test space W_2^*. The W_2 is the standard conforming space of piecewise linear functions over the triangulation $\mathcal{T}_2 = \{T\}$ of Ω_2 into triangles in 2-D and tetrahedra in 3-D (we call them simplices). To introduce the test space W_2^* we need a dual partition $\mathcal{V}_2$ of the domain into the finite (control) volumes V. Let $\mathcal{N}$ denote the set of all vertices (nodes) of the triangles/tetrahedra from $\mathcal{T}_2$ and let $\mathcal{N}_0$ be a subset of those vertices that are not on the Dirichlet part of the boundary $\partial\Omega_2 \setminus \Gamma$. In each simplex $T \in \mathcal{T}_2$ one selects an interior node x_T. Next, in 2-D one links x_T with the midpoints of the sides of the triangle. In this way the triangle is split into three quadrilaterals. In 2-D, one can select x_T to be the orthocenter of the finite element T and then the edges of the volume $V(x)$ will be the perpendicular bisectors of the finite element edges (see the right Figure 2). With each vertex $x \in \mathcal{N}$ of a simplex from $\mathcal{T}_2$, we associate a volume $V = V(x)$ that consists of all quadrilateral/polyhedra having x as a vertex (see Figure 2 for finite volumes in 2-D). The splitting of Ω_2 into finite volumes V forms the partition $\mathcal{V}_2$ (see, Figure 3).

Consider now the test space W_2^* spanned by the characteristic functions of the volumes $V \in \mathcal{V}_2$ and that vanish at the nodes $\mathcal{N} \setminus \mathcal{N}_0$ on the boundary $\partial\Omega_2 \setminus \Gamma$. If one defines the piecewise constant interpolant I_h^* with respect to the volumes $V \in \mathcal{V}_2$, then the space W_2^* is actually equal to $I_h^* W_2$ because they have the same degrees of freedom (associated with the vertices $x \in \mathcal{N}$).

The $L^2(\Omega_2)$ and $H^1(\Omega_2)$ norms in W_2 are defined in a standard way. We shall need also discrete variants of these norms for functions in W_2^*. First, we define the interpolation operator $I_h : W_h^* \longmapsto W_h$ by the following natural rule: $I_h v_2^*$ is the piece-wise linear interpolant of v_2^* over each finite element $T \in \mathcal{T}_2$. Then we define $\|v_2^*\|_{1,h} = \|I_h v_2^*\|_{1,\Omega_2}$. This norm is essentially formed by the squared differences of the values of v_2^* at the vertices of each finite element.

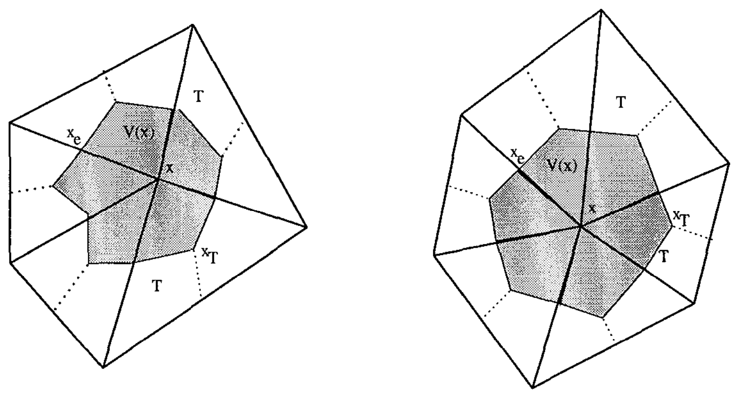

Figure 2: A finite volume V associated with a vertex from the primal triangulation. Left: the vertices of V interior to the triangles T are arbitrary, whereas those on the edges of T are midpoints.

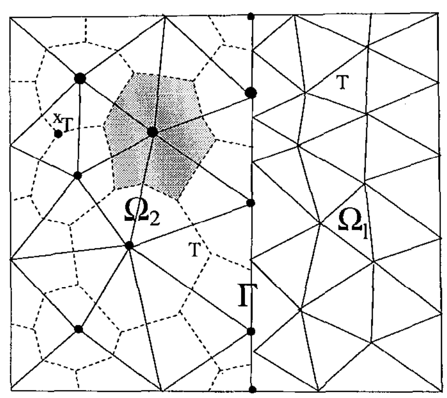

Figure 3: In Ω_1 we use the lowest order Raviart-Thomas spaces over the finite elements T; in Ω_2 we use a solution space W_2 of continuous functions that are linear over the finite elements T and a test space W_2^* of pice-wise constant functions over the volumes V.

Next, we define the following, in general nonsymmetric, bilinear form on $W_2 \times W_2^*$:

$$\begin{aligned} a_{2,h}(v_2, w_2^*) &= -\sum_{x\in\mathcal{N}_0} w_2^*(x) \int\limits_{\partial V(x)} a\nabla p_{2,h}\cdot \mathbf{n}\, ds \\ &= -\sum_{x\in\mathcal{N}_0} \int\limits_{\partial V(x)} a\nabla p_{2,h}\cdot \mathbf{n}\, w_2^*\, ds, \text{ for all } v_2 \in W_2,\ w_2^* \in W_2^*. \end{aligned}$$

Then the finite volume approximation of (15) is: find $v_2 \in W_2$ which satisfies the following identity for all $w_2^* \in W_2^*$;

$$a_{2,h}(v_2, w_2^*) = (f, w_2^*) - \int\limits_{\Gamma} \eta_N w_2^* ds. \tag{16}$$

We note that the integrals over ∂V for $V = V(x)$ a volume corresponding to a Neumann node x (i.e., if $x \in \Gamma$), contain only the interior (to Ω_2) part of ∂V.

We assume that the triangulation $\mathcal{T}_2$ is aligned with the possible jumps of the coefficient matrix $a(x)$, i.e. over each finite element $T \in \mathcal{T}_2$ the matrix $a(x)$ has smooth elements. Therefore, there is a constant $C_0 > 0$ such that for all $T \in \mathcal{T}_2$

$$\begin{gathered} -C_0 h\tilde{a}(x) \le a(x) - \tilde{a}(x) \le C_0 h\tilde{a}(x), \\ \text{where } \tilde{a}(x) = \int_T a(s)ds/\text{meas}(T),\ x \in T. \end{gathered} \tag{17}$$

These inequalities of two $d \times d$ matrices with real elements are understood in the sense of inequalities for the corresponding bilinear forms, i.e. $a \ge \tilde{a}(x)$, iff $\xi^T a\xi \ge \xi^T \tilde{a}(x)\xi$, $\forall\, \xi \in \mathcal{R}^d$. Also, the above equality of the matrices $a(x)$ and $\tilde{a}(x)$ is understood in element-by-element sense, i.e. the elements of $\tilde{a}(x)$ are the mean values over T of the corresponding elements of $a(x)$. Obviously, in case of piece-wise constant coefficients $a(x) \equiv \tilde{a}(x)$ and $C_0 = 0$.

The well posed-ness of the finite volume element approximation follows from the weak coercivity of the bilinear from $a_{2,h}(v_2, w_2^*)$ for sufficiently fine partitions $\mathcal{T}_2$. We have:

Lemma 3 *Let the partition $\mathcal{T}_2$ be so fine that $h < 1/C_0$, where the constant C_0 is determined in (17). Then the following inequality holds true*

$$\sup_{w_2^* \in W_2^*} \frac{a_{2,h}(v_2, w_2^*)}{\|w_2^*\|_{1,h}} \ge C\|v_2\|_{1,\Omega_2},\ \textit{for all } v_2 \in W_2$$

with a constant C independent of h.

In Ω_1 we use the lowest order Raviart-Thomas spaces. Thus, $\mathbf{V}$ as a subspace of $H(div;\Omega_1)$, and W_1 is the space of piece-wise constant functions with respect to the partition $\mathcal{T}_1$ and therefore a subspace of $H^1_{loc}(\Omega_1)$ for $\mathcal{K} = \mathcal{T}_1$. Thus, the advection term $\mathcal{C}p_1$ will be discretized using the pressure space W_1 of piecewise constant functions on the triangulation $\mathcal{T}_1$. For the discretization in Ω_2 we will use the space $W_2 \subset H^1_0(\Omega_2,\ \partial\Omega_2 \setminus \Gamma)$ of continuous piecewise linear functions on $\mathcal{T}_2$. Further, in the finite volume setting we use as a test space W_2^* of piecewise constant functions on $\mathcal{V}_2$. Thus, applying equation (4) consecutively for $(v_1,\ w_1) \in W_1 \times W_2$, and $(v_2, w_2^*) \in W_2 \times W_2^*$, respectively, we get the mixed finite element and finite volume approximations, respectively, of the bilinear form corresponding to the first order term. However, like in the standard Galerkin finite element method this approximation of the operator $\mathcal{C}$ will lead to central differencing, which in turn will lead to a conditionally stable (only for sufficiently small step-size h) scheme. In order to derive a unconditionally stable scheme we shall use upwind approximation in Ω_2.

3.2. Derivation of the coupled method

Since both v_1 and w_1 are discontinuous piece-wise constant functions with respect to the triangulation $\mathcal{T}_1$ the formula (4) is applied in a straightforward manner for $\mathcal{K} = \mathcal{T}_1$ so we get the following approximations a^h_{11} and a^h_{12} of the forms a_{11} and a_{12}, respectively:

$$\begin{aligned} a^h_{11}(v_1, w_1) = \sum_{T\in\mathcal{T}_1} \int_{\partial T\setminus\Gamma_-} & \left[(\underline{b}\cdot\mathbf{n})_-\ v_1^o + (\underline{b}\cdot\mathbf{n})_+\ v_1^i\right] w_1^i\ ds \\ & +(c_0 v_1,\ w_1) \text{ for } \quad v_1 \in W_1,\ w_1 \in W_1 \end{aligned} \tag{18}$$

and

$$a^h_{12}(v_2, w_1) = \int_{\Gamma_-} I_h^* v_2\ w_1\ \underline{b}\cdot\mathbf{n}_1\ ds \text{ for } \quad v_2 \in W_2,\ w_1 \in W_1. \tag{19}$$

Now we find the contributions of the the operator $\mathcal{C}$ from Ω_2 and we define the approximations of the bilinear forms a_{21} and a_{22}. We shall simply rewrite (4) for $\mathcal{K} = \mathcal{V}_2$:

$$\begin{aligned} C(v_2, w_2^*) = \sum_{V\in\mathcal{V}_2} \Big(& -\int_V v_2\ \underline{b}\cdot\nabla w_2^*\ dx + \int_{\partial V_-} v_2^o\ w_2^{*i}\ \underline{b}\cdot\mathbf{n}\ ds \\ & + \int_V c_0(x)\ v_2\ w_2^* dx + \int_{\partial V_+} v_2^i\ w_2^{*i}\ \underline{b}\cdot\mathbf{n}\ ds\Big). \end{aligned} \tag{20}$$

Since the functions in W_2 are continuous then $C(v_2, w_2^*)$ is well defined for all $v_2 \in W_2$ and $w_2^* \in W_2^*$. Taking into account that the functions in W_2^* are

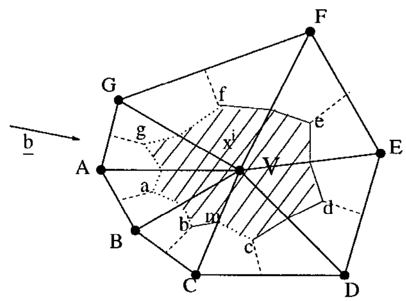

Figure 4: The shaded area is the volume V centered at the vertex x^i; the doted boundary of V denotes the "inflow" boundary, while the solid one is the "outflow" part; on the various pieces of the boundary ∂V we have the following approximation of the convection term: on (a, b), (m, c), (f, g) and (g, a): $v_2(x) = v_2(x^i)$; on (b, m) : $v_2(x) = v_2(x^o) = v_2(C)$; on (c, d) : $v_2(x) = v_2(x^o) = v_2(D)$; on (d, e) : $v_2(x) = v_2(x^o) = v_2(E)$; on (e, f) : $v_2(x) = v_2(x^o) = v_2(F)$.

constant over each finite volume $V \in \mathcal{V}_2$ then the contributions from each finite volume $V \in \mathcal{V}_2$ are:

$$\int_{\partial V} \left[(\underline{b} \cdot \mathbf{n})_- v_2^o + (\underline{b} \cdot \mathbf{n})_+ v_2^i \right] \ w_2^{*i} ds + \int_V c_0 \ v_2 \ w_2^* dx. \tag{21}$$

Since v_2 is continuous then obviously, we have $v_2^o = v_2^i = v_2(x)$. On the boundary Γ_+ the values v_2^o are not defined (this is the inflow boundary for Ω_2) and we shall take them from the corresponding counterpart in Ω_1, i.e. as $v_1(x)$. Thus, we split the integrals over ∂V into two parts and get

$$\int_{\partial V \setminus \Gamma_+} \underline{b} \cdot \mathbf{n} \ v_2 w_2^{*i} ds + \int_{\partial V \cap \Gamma_+} \underline{b} \cdot \mathbf{n} \ v_1 w_2^{*i} ds.$$

Unfortunately, the exact calculation of the first integral in (21) will lead to central differences and therefore to a scheme which is stable only for sufficiently small step-size h. The limitation of the step-size h will depend on the magnitude of the convection coefficient $\underline{b}$ relative to the diffusion coefficient (matrix) a. For problems with dominating convection this will lead to prohibitively small h. In order to avoid this conditional stability we introduce an up-wind approximation of the integrals. This approximation is done in the following way. We denote by

$V(x^i)$ a finite volume centered at the vertex x^i and by $V(x^o)$ any of neighboring volumes centered at the vertices x^o. The integral over $\partial V \setminus \Gamma_-$ is split into sub-integrals over the boundaries of V $\gamma = \overline{V(x^i)} \cap \overline{V(x^o)} \cap T$ with its neighboring finite (control) volumes and contained in the finite element T. We assume that over each γ the function $\underline{b}(x) \cdot \mathbf{n}$ does not change sign (i.e. is either nonnegative or negative). Then on γ we use upwind approximation of the following form (for a 2-D illustration, see Figure 4):

$$\underline{b} \cdot \mathbf{n}\, v_2(x) \approx (\underline{b} \cdot \mathbf{n})_+ v_2(x^i) + (\underline{b} \cdot \mathbf{n})_- v_2(x^o), \quad \text{for} \quad x \in \gamma.$$

Note, that in the finite volume $V(x^i)$ we have $I_h^* v_2(x) = v_2(x^i)$. Similar equalities are valid for the neighboring volumes $V(x^o)$ as well. Thus, roughly speaking the function $v_2(x)$ has been replaced by its interpolant in the space of discontinuous functions W_2^* and then taken the appropriate (up-wind or in the opposit direction of the vector-field $\underline{b}(x)$) values at the finite volume interfaces. A particular finite volume in 2-D is shown on Figure 4.

Summing for all $V \in \mathcal{V}_2$ we finally get the following form by taking also into account the diffusion term (16):

$$\begin{aligned} a_{22}^h(v_2,\, w_2^*) \;&= \sum_{V \in \mathcal{V}_2} \int_{\partial V \setminus \Gamma_+} \left[(\underline{b} \cdot \mathbf{n})_+ v_2(x^i) + (\underline{b} \cdot \mathbf{n})_- v_2(x^o) \right] w_2^{*i}\, ds \\ &\qquad + (c_0 I_h^* v_2,\, w_2^*) + a_{2,h}(v_2,\, w_2^*), \end{aligned} \tag{22}$$

for all $v_2 \in W_2$, $w_2^* \in W_2^*$ and the form

$$a_{21}^h(v_1,\, w_2^*) = \int_{\Gamma_+} v_1\, w_2^*\, \underline{b} \cdot \mathbf{n}_2\, ds \quad \text{for} \quad v_1 \in W_1, \;\; w_2^* \in W_2^*. \tag{23}$$

The coupled mixed finite element/finite volume approximation of the composite problem (12) reads as: find $\mathbf{u}_h \in \mathbf{V}$, $p_{1,h} \in W_1$, and $p_{2,h} \in W_2$, such that

$$\begin{array}{llll} (a^{-1}\mathbf{u}_h, \mathbf{v}) & -(p_{1,h}, \nabla \cdot \mathbf{v}) & + < p_{2,h}, \mathbf{v} \cdot \mathbf{n}_1 >_\Gamma = 0, & \\ -(\nabla \cdot \mathbf{u}_h,\, w_1) & -a_{11}^h(p_{1,h},\, w_1) & -a_{12}^h(I_h^* p_{2,h},\, w_1) = -(f, w_1), & \qquad (24) \\ < \mathbf{u}_h \cdot \mathbf{n}_1,\, I_h w_2^* >_\Gamma & -a_{21}^h(p_{1,h},\, w_2^*) & -a_{22}^h(p_{2,h}, w_2^*) = -(f, w_2^*) & \end{array}$$

for all $\mathbf{v} \in \mathbf{V}$, $w_1 \in W_1$, and $w_2^* \in W_2^*$, respectively.

3.3. Stability of the coupled scheme and error estimate

An important feature of the described above discretizations is that the corresponding operator is coercive in an appropriate norm and the method is stable.

For proving the stability we shall follow the same argument as in the case of original setting. Let as before $\mathcal{E} = \{e\}$ be the set of edges/faces of the elements from $\mathcal{T}_1$ and let $\mathcal{E}_0$ be the set of interior edges/faces. Similarly, $\mathcal{G} = \{\gamma\}$ is the set of edges/faces, with each γ being the boundary of two adjacent volumes $V_1 \in \mathcal{V}_2$ and $V_2 \in \mathcal{V}_2$ contained in a finite element T, i.e., $\gamma = \overline{V}_1 \cap \overline{V}_2 \cap T$. This splitting can be also used in the computational procedure, since it will lead to element-wise contributions of the convection term to the stiffness matrix. Note, that all edges/faces γ are in the interior of Ω_2. For the coercivity of the coupled problem we need the following discrete variant of the norm (13):

$$\begin{aligned}\|v_1\|^2_{*,h,\Omega_1} \quad &+\|v_2^*\|^2_{*,h,\Omega_2} = \frac{1}{2}\sum_{\gamma\in\mathcal{G}} \int_\gamma [v_1]^2 |\underline{b}\cdot\mathbf{n}|\, ds + \gamma_0 \|v_1\|^2_{0,\Omega_1} \\ &+\frac{1}{2}\int_{\partial\Omega_{1+}\setminus\Gamma_+} v_1^2\, \underline{b}\cdot\mathbf{n}_1\, ds - \frac{1}{2}\int_{\partial\Omega_{1-}\setminus\Gamma_-} v_1^2\, \underline{b}\cdot\mathbf{n}_1\, ds \\ &+\frac{1}{2}\int_{\Gamma_+} (v_1 - v_2^*)^2 \underline{b}\cdot\mathbf{n}_1\, ds - \frac{1}{2}\int_{\Gamma_-} (v_1 - v_2^*)^2 \underline{b}\cdot\mathbf{n}_1\, ds \\ &+\gamma_0\|v_2^*\|^2_{0,\Omega_2} + (\tilde{a}\nabla v_2, \nabla I_h^* v_2).\end{aligned} \tag{25}$$

Here $\tilde{a}$ is a piece-wise constant matrix with respect to the partition $\mathcal{T}_2$ defined by (17).

Theorem 2 *Let $h < 1/C_0$, where C_0 is defined by (17). Then the solution of the problem (12) satisfies the a priori estimate:*

$$\|\mathbf{u}_h\|_{L^2(\Omega_1)} + \|p_{1,h}\|_{*,h,\,\Omega_1} + \|p_{2,h}\|_{*,h,\,\Omega_2} \le C\|f\| \tag{26}$$

where the $(*,h)$-norm is defined by (25) with respect to the partitions $\mathcal{T}_1$ and $\mathcal{T}_2$.

Proof: As in the continuous case, by testing (24) with $\mathbf{v} = \mathbf{u}_h$, $w_1 = -p_{1,h}$, and $w_2 = -I_h^* p_{2,h}$ we get the equation:

$$\begin{aligned}&(a^{-1}\mathbf{u}_h,\ \mathbf{u}_h) + a_{11}^h(p_{1,h},\ p_{1,h}) + a_{12}^h(p_{1,h},\ I_h^* p_{2,h}) \\ &+a_{21}^h(I_h^* p_{2,h},\ p_{1,h}) + a_{22}^h(p_{2,h},\ I_h^* p_{2,h}) = (f,\ p_{1,h}) + (f,\ I_h^* p_{2,h}).\end{aligned} \tag{27}$$

Further, the estimate (14) is a consequence of the simplified form (28) of a_{11}^h and (29) of a_{22}^h, which are established in the lemmas below.

Lemma 4 *For any edge/face e denote by $\mathbf{n}_e$ a fixed unit vector normal to e and let T_e^+ and T_e^- be the two adjacent elements to e. Similarly, for any* *edge/face $\gamma \in \mathcal{G}$ denote by $\mathbf{n}_\gamma$ a unit vector normal to γ pointing to one of the neighboring volumes V_γ^+ and V_γ^-. Also, $[v_1]$ denotes the jump of a function across an underlined boundary (here we have either e or γ) and $\overline{w}_1$ denotes the arithmetic mean of the jump (introduced in Section). Then,*

$$
\begin{aligned}
a_{11}^h(v_1, w_1) \quad &= \frac{1}{2} \sum_{e \in \mathcal{E}_0} \int_e [v_1][w_1] |\underline{b} \cdot \mathbf{n}| \, ds + \sum_{e \in \mathcal{E}_0} \int_e \underline{b} \cdot \mathbf{n}_e [v_1] \, \overline{w}_1 ds \\
&+ (c_0 v_1, \; w_1) + \int\limits_{\Omega_{1+}} \underline{b} \cdot \mathbf{n}_1 \, v_1 w_1 \, ds \quad \text{for } v_1, \; w_1 \in W_1.
\end{aligned} \tag{28}
$$

Similarly, for all $v_2 \in W_2, \; w_2^* \in W_2^*$ the following identity is valid (to simplify the expressions we have used the notation $v_2^* = I_h^* v_2$):

$$
\begin{aligned}
a_{22}^h(v_2, w_2^*) \quad &= \frac{1}{2} \sum_{\gamma \in \mathcal{G}} \int_\gamma [v_2^*][w_2^*] |\underline{b} \cdot \mathbf{n}| \, ds + \sum_{\gamma \in \mathcal{G}} \int_\gamma \underline{b} \cdot \mathbf{n}_\gamma \, [v_2^*] \, \overline{v}_2^* \, ds \\
&+ (c_0 v_2^*, \; w_2^*) - \int\limits_{\Gamma_-} \underline{b} \cdot \mathbf{n}_1 \, v_2^* w_2^* \, ds \; + a_{2,h}(v_2, w_2^*).
\end{aligned} \tag{29}
$$

Proof: The proof of (28) and (29) essentially repeats the arguments of Lemma 1. There is a small difference in the proof of (29) where the integrals over each $\gamma \in G$ have been computed by using up-wind approximation.

Lemma 5 *The following identity is valid for all $v_1 \in W_1$:*

$$
\begin{aligned}
a_{11}^h(v_1, v_1) = \quad &\frac{1}{2} \sum_{e \in \mathcal{E}_0} \int_e [v_1]^2 \, |\underline{b} \cdot \mathbf{n}_1| \, ds + ((c_0 + \frac{1}{2} \nabla \cdot \underline{b}) v_1, \; v_1) \\
&+ \frac{1}{2} \int_{\partial \Omega_{1+}} v_1^2 \, \underline{b} \cdot \mathbf{n}_1 \, ds - \frac{1}{2} \int_{\partial \Omega_{1-}} v_1^2 \, \underline{b} \cdot \mathbf{n}_1 \, ds.
\end{aligned} \tag{30}
$$

Similarly, for all $v_2 \in W_2$ (here in order to simplify we use the notation $v_2^* = I_h^* v_2$):

$$
\begin{aligned}
a_{22}^h(v_2, v_2^*) = \quad &\frac{1}{2} \sum_{\gamma \in \mathcal{G}} \int_\gamma [v_2^*]^2 \, |\underline{b} \cdot \mathbf{n}_1| \, ds + ((c_0 + \frac{1}{2} \nabla \cdot \underline{b}) v_2^*, \; v_2^*) \\
&+ \frac{1}{2} \int_{\Gamma_+} (v_2^*)^2 \, \underline{b} \cdot \mathbf{n}_1 \, ds - \frac{1}{2} \int_{\Gamma_-} (v_2^*)^2 \, \underline{b} \cdot \mathbf{n}_1 \, ds.
\end{aligned} \tag{31}
$$

Finally, we have the following error estimate:

Theorem 3 *Assume that the solution $p(x)$ of the problem (1) is H^2-regular in Ω. Then the solution $(\mathbf{u}_h, p_{1,h}, p_{2,h})$ of the coupled mixed discontinuous finite element and finite volume methods (24) converges to the solution $(\mathbf{u}, p_1, p_2)$ of the composite problem (12) and the following error estimate holds true:*

$$\|\mathbf{u} - \mathbf{u}_h\| + \|p_1 - p_{1,h}\|_{*,\Omega_1} + \|p_2 - p_{2,h}\|_{*,\Omega_2} \leq C(h_1^{\frac{1}{2}} + h_2)\|p\|_{2,\Omega}. \tag{32}$$

The constant C does not depend on h but may depend on the ratios $\frac{h_1}{h_2}$ and $\frac{h_2}{h_1}$.

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