UCRL-ID-137122

The Electrostatic Field Between Non-Concentric Cylinders

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January 10, 2000

U.S. Department of Energy

Lawrence
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This work was performed under the auspices of the U. S. Department of Energy by the University of California, Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.

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This report describes a closed-form solution to the electrostatic potential, and the electric field, between non-concentric cylinders, with the inner cylinder charged and the outer cylinder grounded. This problem is an abstraction of the situation of an electron beam within a drift tube. Capacitive and surface current probes on the inner wall of the outer cylinder are used to detect the asymmetry of the field when the beam is off center. The solution of this problem allows for a quantitative relationship between probe-array signals and beam deflection. Probe-arrays of this type are called "beam bugs" at LLNL. This problem was brought to my attention by the work of Tom Fessenden as described in his reports [1] and [2]. **Figure 13** shows a schematic of the problem.

The solution described here is suggested by the analysis presented in [3]. The essential point is that the 2D potential for a line source decreases along a radius as the logarithm of the distance. The non-concentric cylinder problem has a unique profile of this type for each ray from (ρ, σ) linking the inner cylinder at equipotential V_2, and the outer cylinder at equipotential 0.

The solution will be illustrated with a specific example. Referring to Figure 13, the following parameters are chosen:

$V_2 := 1$	voltage of the inner cylinder
$b := 1$	radius of the inner cylinder
$r_1 := 2.3$	radius of the outer cylinder
$\rho := -0.96$	x coordinate of the center of the inner cylinder
$\sigma := 0.37$	y coordinate of the center of the inner cylinder

The grid for Figures 1 through 12 in this report is set below.

$Nx := 80$ number of points along the x axis, index $i = 0, 1,..., N_x - 1$

$Ny := 80$ number of points along the y axis, index $j = 0, 1,..., N_y - 1$

$$i := 0, 1 .. Nx - 1$$

$$\Delta x := \frac{2.5}{2.3} \cdot 2 \cdot \frac{r_1}{Nx - 1}$$ x increment

$$x_i := -r_1 \cdot \frac{2.5}{2.3} + i \cdot \Delta x$$ x points, centered at $x = 0$

$$j := 0, 1 .. Ny - 1$$

$$\Delta y := \frac{2.5}{2.3} \cdot 2 \cdot \frac{r_1}{Ny - 1}$$ y increment

$$y_j := -r_1 \cdot \frac{2.5}{2.3} + j \cdot \Delta y$$ y points, centered at $y = 0$.

The radial coordinate [for each (i, j) pair] is

$$r_{i,j} := \sqrt{(x_i)^2 + (y_j)^2}$$

and the normalized radial extent with respect to the inner cylinder is

$$R_{i,j} := \frac{1}{b} \cdot \sqrt{\left(x_i - \rho\right)^2 + \left(y_j - \sigma\right)^2}$$

The normalized radial extent to the outer cylinder, with respect to the inner cylinder, and which intersects point (x, y) is

$$T_{i,j} := \frac{r_1}{b} \cdot \sqrt{\left[\frac{x_i}{\sqrt{(x_i)^2 + (y_j)^2}} - \frac{\rho}{r_1}\right]^2 + \left[\frac{y_j}{\sqrt{(x_i)^2 + (y_j)^2}} - \frac{\sigma}{r_1}\right]^2}$$

The potential at each grid point (i, j) is

$$V_{i,j} := V_2 \cdot \left(1 - \frac{\ln(R_{i,j})}{\ln(T_{i,j})} \cdot \Phi(R_{i,j} - 1)\right) \cdot \Phi\left(1 - \frac{r_{i,j}}{r_1}\right)$$

where the Φ are unit step functions centered at 0. The step functions cut off the solution within the inner cylinder and beyond the outer one. It is convenient to define a negative potential, for display purposes.

$$Vn_{i,j} := -V_{i,j}$$

The electric field is **E** = -grad(V), and the cartesian and cylindrical components of this field are defined in the following way. First, the partial derivatives of R with respect to x and y are shown as R_x and R_y respectively:

$$Rx_{i,j} := \frac{x_i - \rho}{b^2 \cdot R_{i,j}} \qquad \text{and} \qquad Ry_{i,j} := \frac{y_j - \sigma}{b^2 \cdot R_{i,j}}$$

Also, the partials of T with respect to x and y are T_x and T_y, respectively:

$$Tx_{i,j} := \frac{r_1}{b^2\cdot(T_{i,j})^2\cdot r_{i,j}}\cdot\left[\left(\frac{x_i}{r_{i,j}}-\frac{\rho}{r_1}\right)\cdot\left[1-\left(\frac{x_i}{r_{i,j}}\right)^2\right]\ \ldots \atop + -\left(\frac{y_j}{r_{i,j}}-\frac{\sigma}{r_1}\right)\cdot\left(\frac{y_j}{r_{i,j}}\right)^2\right]$$

$$Ty_{i,j} := \frac{r_1}{b^2\cdot(T_{i,j})^2\cdot r_{i,j}}\cdot\left[\left(\frac{y_j}{r_{i,j}}-\frac{\sigma}{r_1}\right)\cdot\left[1-\left(\frac{y_j}{r_{i,j}}\right)^2\right]\ \ldots \atop + -\left(\frac{x_i}{r_{i,j}}-\frac{\rho}{r_1}\right)\cdot\left(\frac{x_i}{r_{i,j}}\right)^2\right]$$

Now the cartesian components of the electric field can be given:

$$Ex_{i,j} := -\frac{-V_2}{\ln(T_{i,j})}\cdot\left(\frac{Rx_{i,j}}{R_{i,j}}-\frac{\ln(R_{i,j})\cdot Tx_{i,j}}{T_{i,j}\cdot\ln(T_{i,j})}\right)\cdot\Phi(R_{i,j}-1)\cdot\Phi\left(1-\frac{r_{i,j}}{r_1}\right)$$

$$Ey_{i,j} := -\frac{-V_2}{\ln(T_{i,j})}\cdot\left(\frac{Ry_{i,j}}{R_{i,j}}-\frac{\ln(R_{i,j})\cdot Ty_{i,j}}{T_{i,j}\cdot\ln(T_{i,j})}\right)\cdot\Phi(R_{i,j}-1)\cdot\Phi\left(1-\frac{r_{i,j}}{r_1}\right)$$

and the negatives of these components (useful for display) defined:

$Exn_{i,j} := -Ex_{i,j}$ and $Eyn_{i,j} := -Ey_{i,j}$

The magnitude of the electric field, which is not necessarily in the radial direction, is given by

$$Em_{i,j} := \sqrt{(Ex_{i,j})^2+(Ey_{i,j})^2}$$

The polar angle, that is to say the direction of the electric field at each point in the (x, y) plane, with respect to center (0, 0) is

$$\theta e_{i,j} := \operatorname{atan}\left(\frac{Ey_{i,j}}{Ex_{i,j}}\right) + \frac{\pi}{2}\cdot\left[1 - \frac{Ex_{i,j}}{\left|Ex_{i,j}\right|} + \left(1 + \frac{Ex_{i,j}}{\left|Ex_{i,j}\right|}\right)\cdot\left(1 - \frac{Ey_{i,j}}{\left|Ey_{i,j}\right|}\right)\right]$$

when the arctangent function is restricted to the range of $-\pi/2$ to $\pi/2$. The angle between 0 and 2π defined by point (x, y) is designated

$$\theta_{i,j} := \operatorname{angle}\left(x_i, y_j\right)$$

The radial and azimuthal components of the field are given, respectively, by:

$$Er_{i,j} := Em_{i,j}\cdot\cos\left(\theta e_{i,j} - \theta_{i,j}\right)$$

$$E\theta_{i,j} := Em_{i,j}\cdot\sin\left(\theta e_{i,j} - \theta_{i,j}\right)$$

The potential V must satisfy Laplace's equation, $\nabla^2 V = 0 = \nabla\cdot\mathbf{E}$, between the cylinders. As it is easy to define and calculate the Laplacian numerically, this will substitute for a result based on the greater effort of differentiating E_x and E_y.

First, the partials of E_x with respect to x at the edges of the grid are defined:

$$dEx_{0,j} := \frac{Ex_{1,j} - Ex_{0,j}}{\Delta x} \quad \text{and} \quad dEx_{Nx-1,j} := \frac{Ex_{Nx-1,j} - Ex_{Nx-2,j}}{\Delta x}$$

Then, the partials of E_y with respect to y at the edges of the grid are defined:

$$dEy_{i,0} := \frac{Ey_{i,1} - Ey_{i,0}}{\Delta y} \quad \text{and} \quad dEy_{i,Ny-1} := \frac{Ey_{i,Ny-1} - Ey_{i,Ny-2}}{\Delta y}$$

Finally, the x partial of E_x, and the y partial of E_y are defined as second order differences within the body of the grid:

$$\text{for} \quad i := 1, 2 .. Nx - 2 \quad \text{and} \quad j := 1, 2 .. Ny - 2$$

$$dEx_{i,j} := \frac{Ex_{i+1,j} - Ex_{i-1,j}}{2 \cdot \Delta x} \quad \text{and} \quad dEy_{i,j} := \frac{Ey_{i,j+1} - Ey_{i,j-1}}{2 \cdot \Delta y}$$

Now the Laplacian is constructed for the grid

$$\text{for} \quad i := 0, 1 .. Nx - 1 \quad \text{and} \quad j := 0, 1 .. Ny - 1$$

$$dE_{i,j} := dEx_{i,j} + dEy_{i,j}$$

The results for this example follow as twelve figures. Both surface and grey (color) scale contour plots are shown for each of:

1 V, potential,
2 V_n, negative of the potential,
3 E_x, x component of the electric field, **E** = -grad(V),
4 E_y, y component of the electric field,
5 E_{xn}, negative of E_x,
6 E_{yn}, negative of E_y,
7 E_m, magnitude of the electric field, $\sqrt{(E_x^2 + E_y^2)}$,
8 E_r, radial field, with respect to center (0, 0),
9 E_θ, azimuthal field, with respect to center (0, 0),
10 θ, a display of spatial phase, the complex plane, note the 0-2π jump along the +x axis (the same orientation for all plots),
11 θ_e, the angle of the local field, note 0-2π jump,
12 $\nabla \cdot$**E**, the Laplacian, the divergence of **E**.

Note that $\nabla \cdot$**E** is zero between the cylinders, as should be the case.

References

[1] T. Fessenden, "Formal solution for the fields within a beam-bug calibrator," AHF-99-1020, LLNL, 13 July 1998

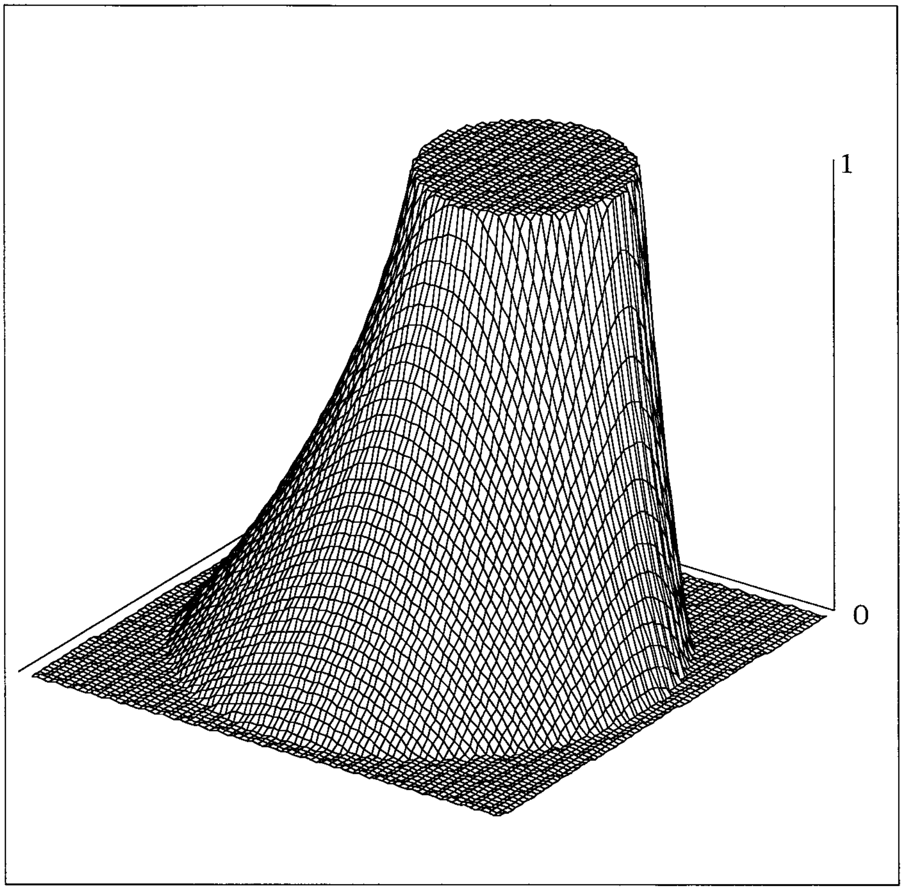
[2] J. C. Clark, T. J. Fessenden, C. Holmes, "Toward more precise beam position measurements," AHF-99-1021, LLNL, 12 May 1999

[3] "The 2D electric field above a planar sequence of independent strip electrodes," UCRL-JC-136248, 4 October 1999, submitted for publication in the *Journal of Physics D: Applied Physics*

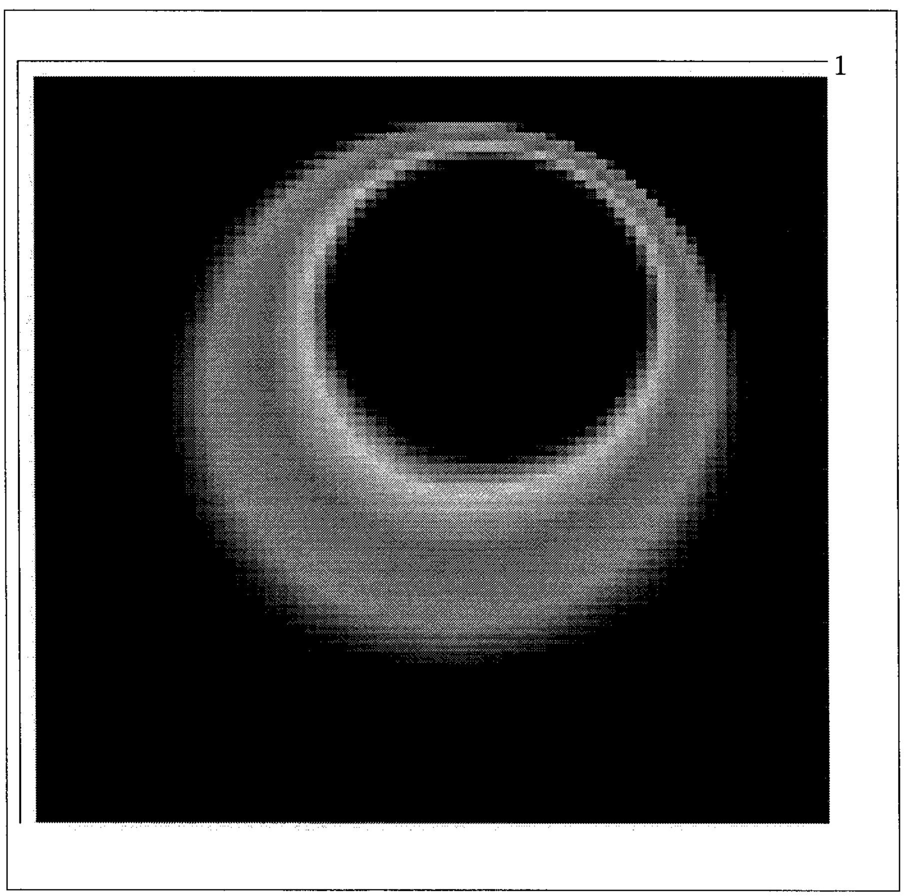
Figure 13 caption

Fessenden's electrostatic problem. A diagram of the geometry, and the nomenclature used. The outer cylinder has radius r_1 and a center at $(x, y) = (0, 0)$. The inner cylinder has radius b and a center at $(x, y) = (\rho, \sigma)$. The radial extent from (0, 0) to point (x, y) is $r(x, y)$, and the radial extent from point (ρ, σ) to point (x, y) is $b^*R(x, y, \rho, \sigma, b)$. R is normalized to 1 at the periphery of the inner cylinder, and is designated by the function $T(x, y, \rho, \sigma, b, r_1)$ for points (x, y) on the periphery of the outer cylinder. The space between the cylinders must satisfy Laplace's equation. The radial field $E_r(@\ r_1)$ at the outer cylinder would be what a capacitive probe senses, while the azimuthal field $E_\theta(@\ r_1)$ would be what a wall current probes senses $[j_\theta(@\ r_1) =$ conductivity $* E_\theta(@\ r_1)]$.

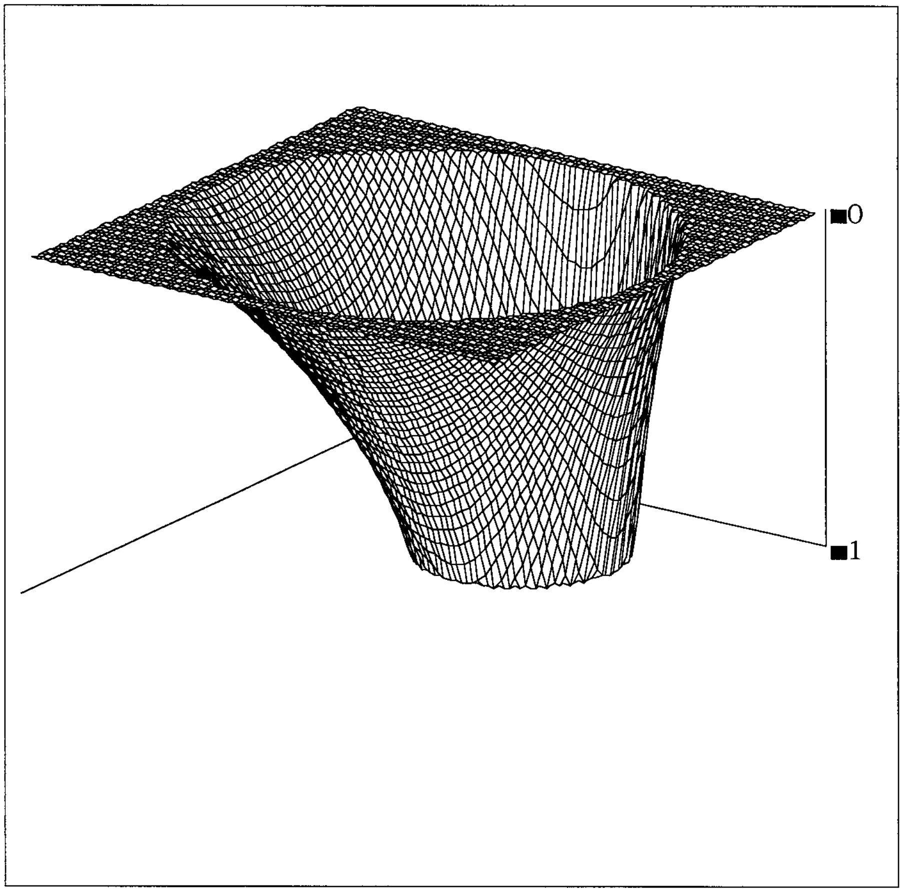
This work was performed under the auspices of the U. S. DOE by LLNL under contract W-7405-Eng-48.



V



V



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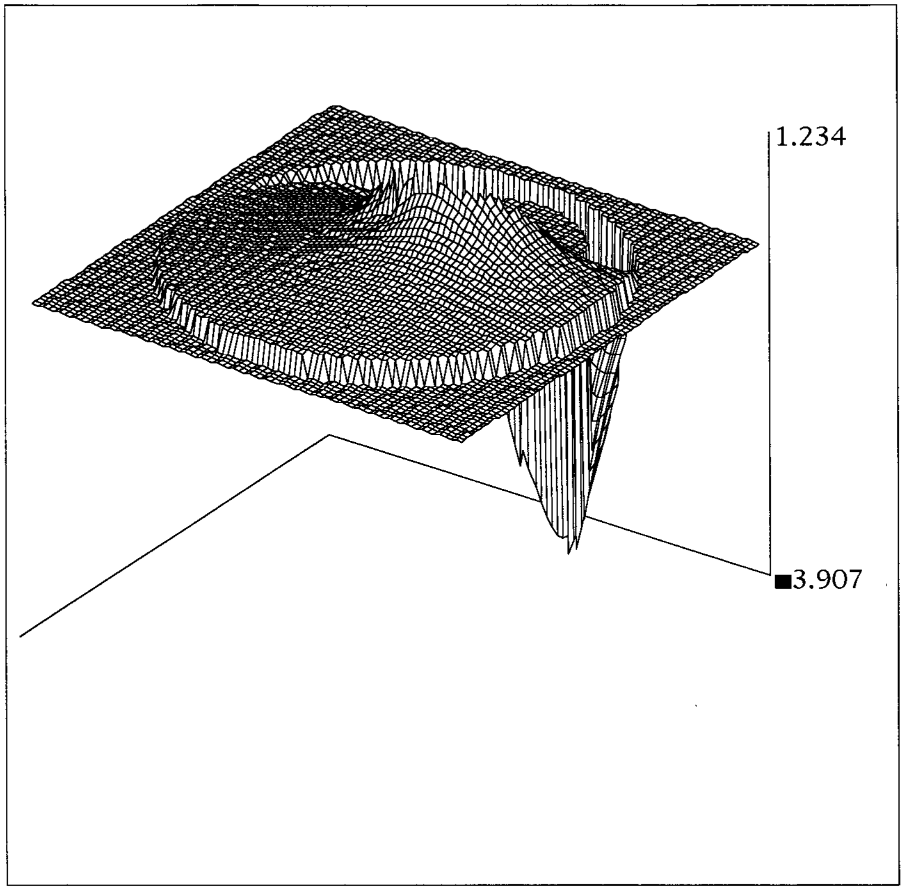
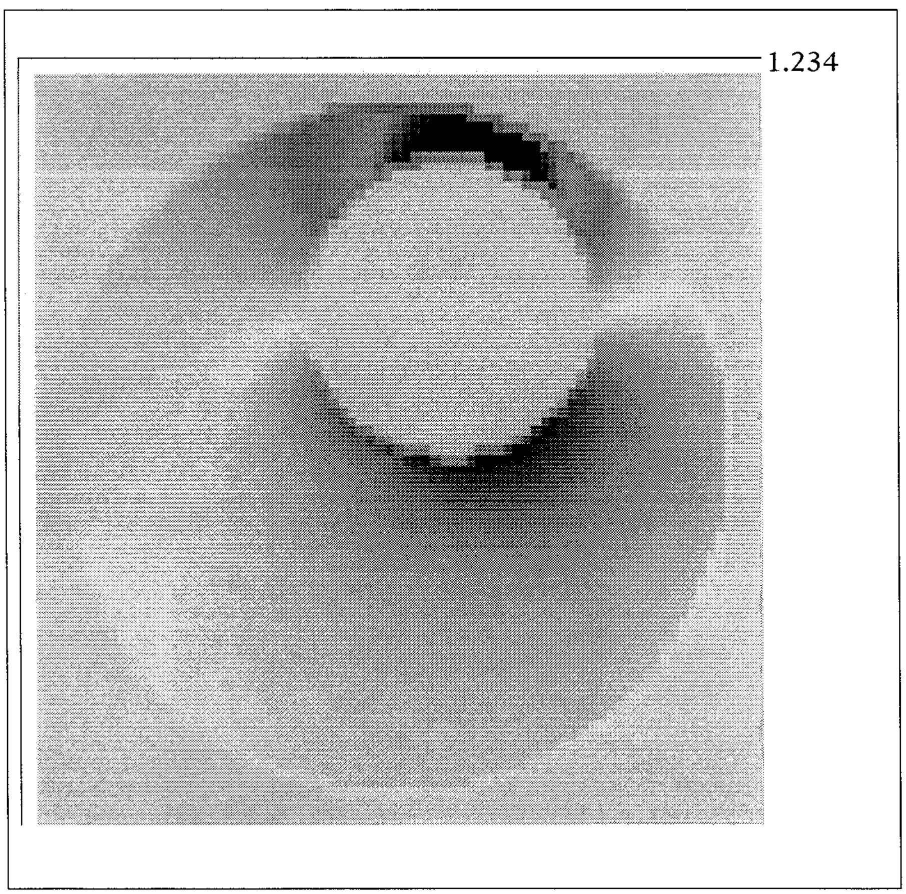


Ex



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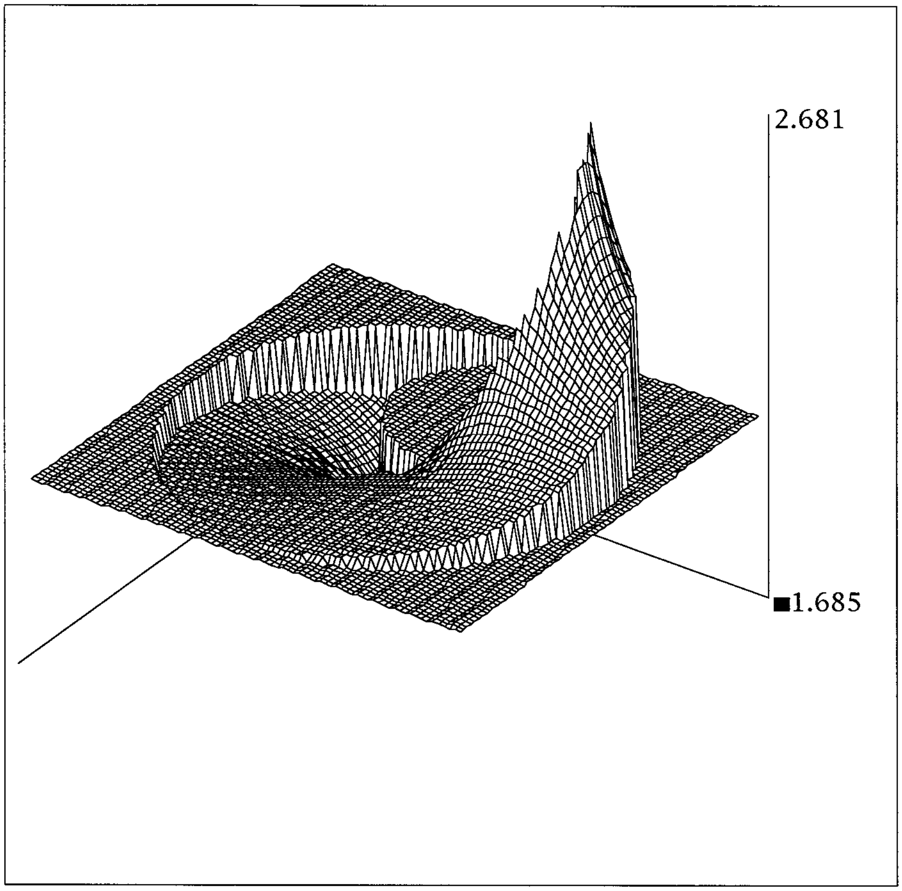
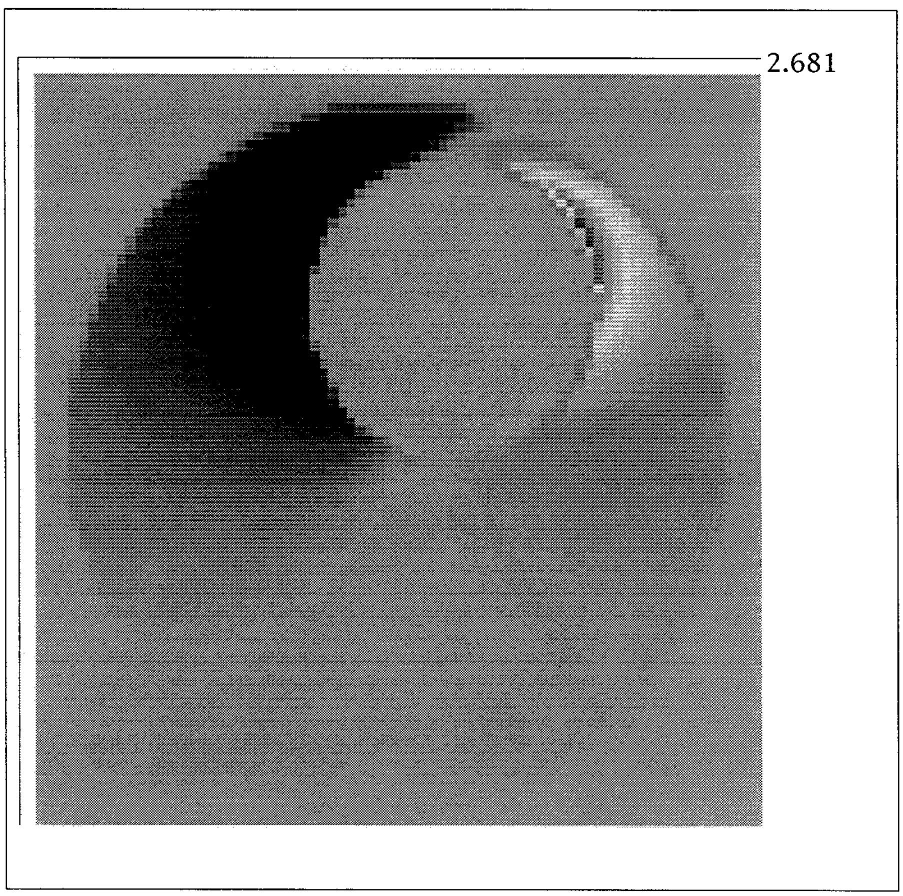


Ey



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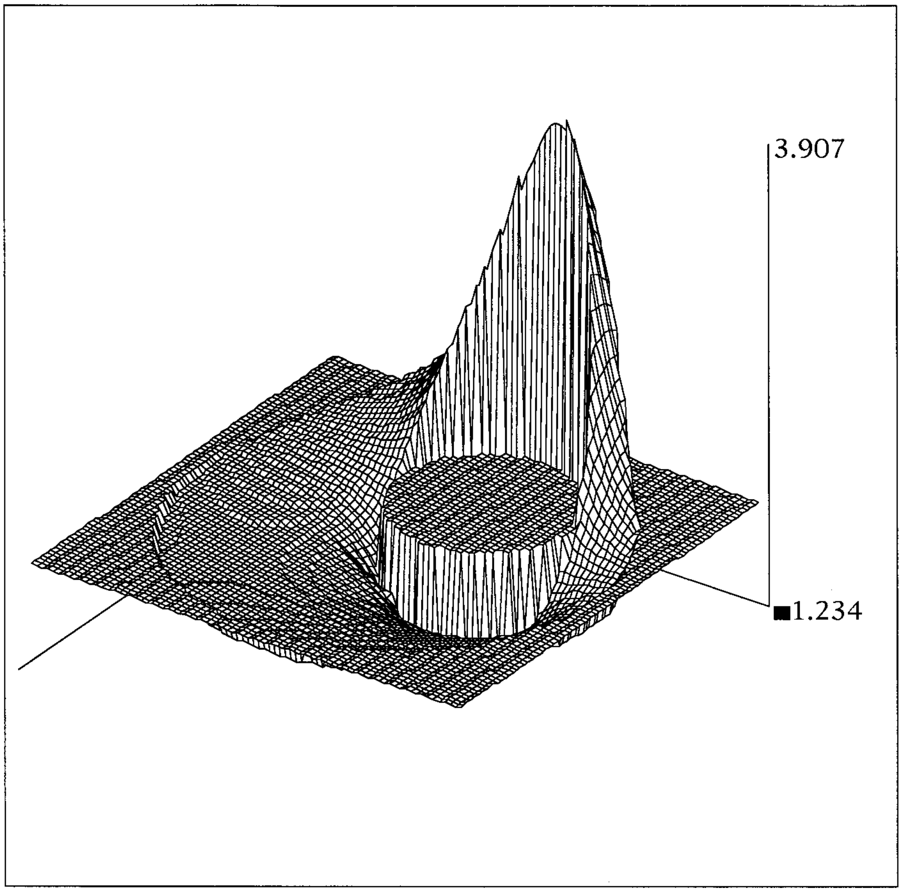

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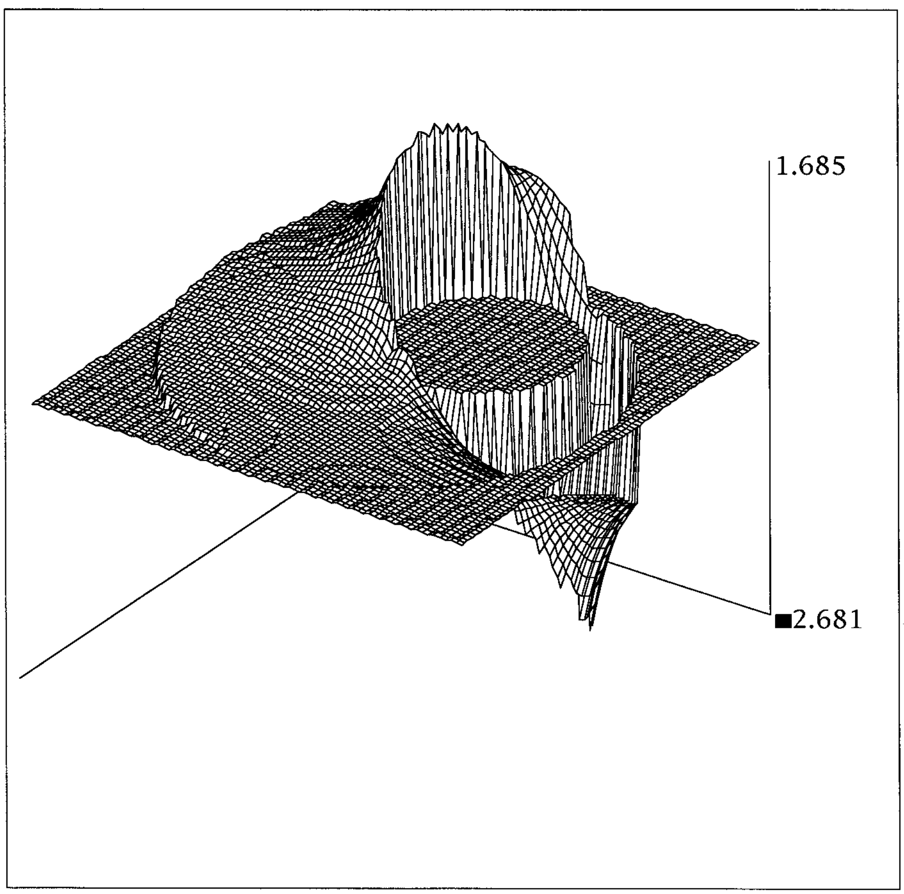

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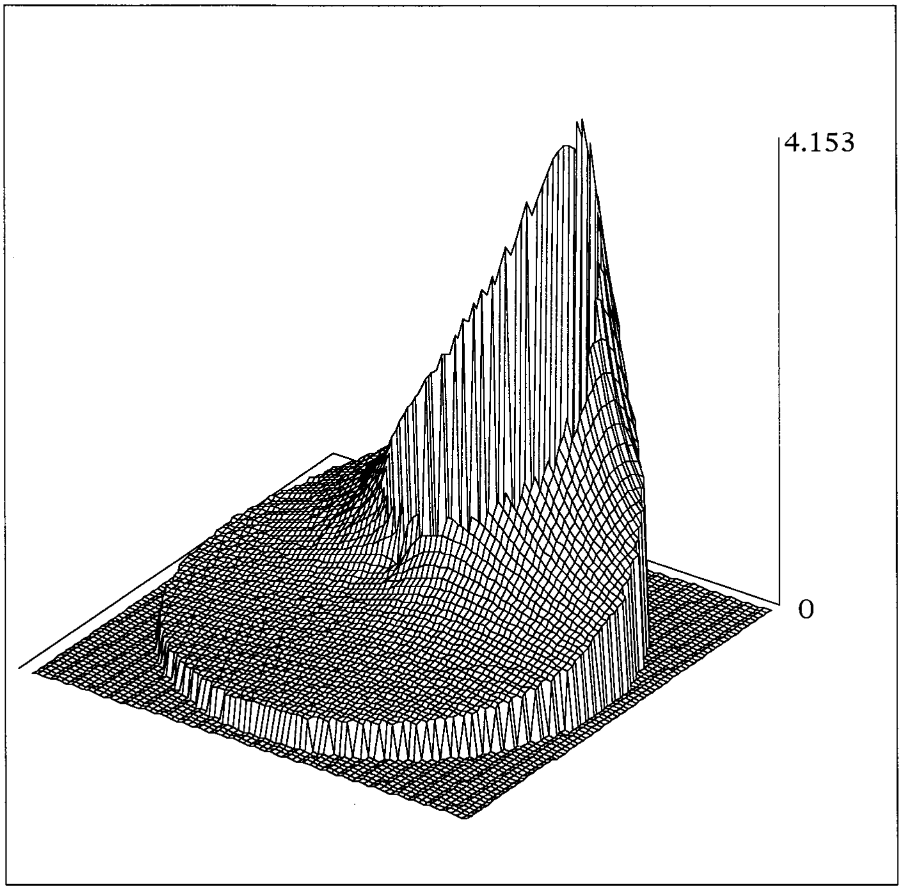
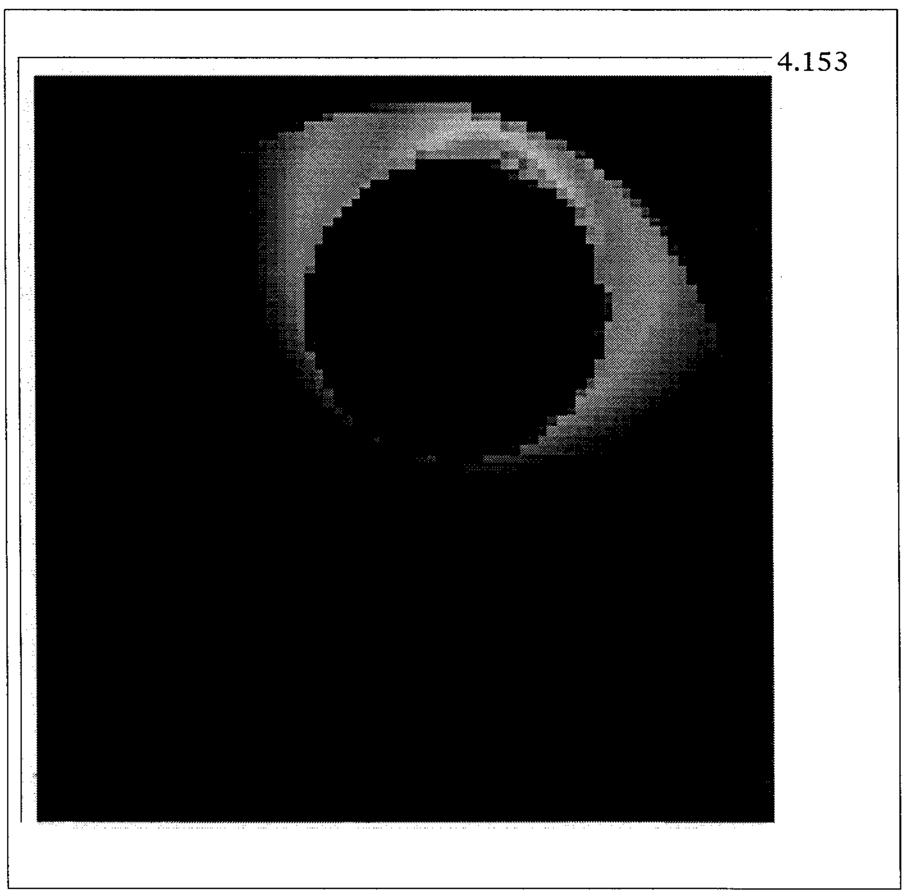


Em



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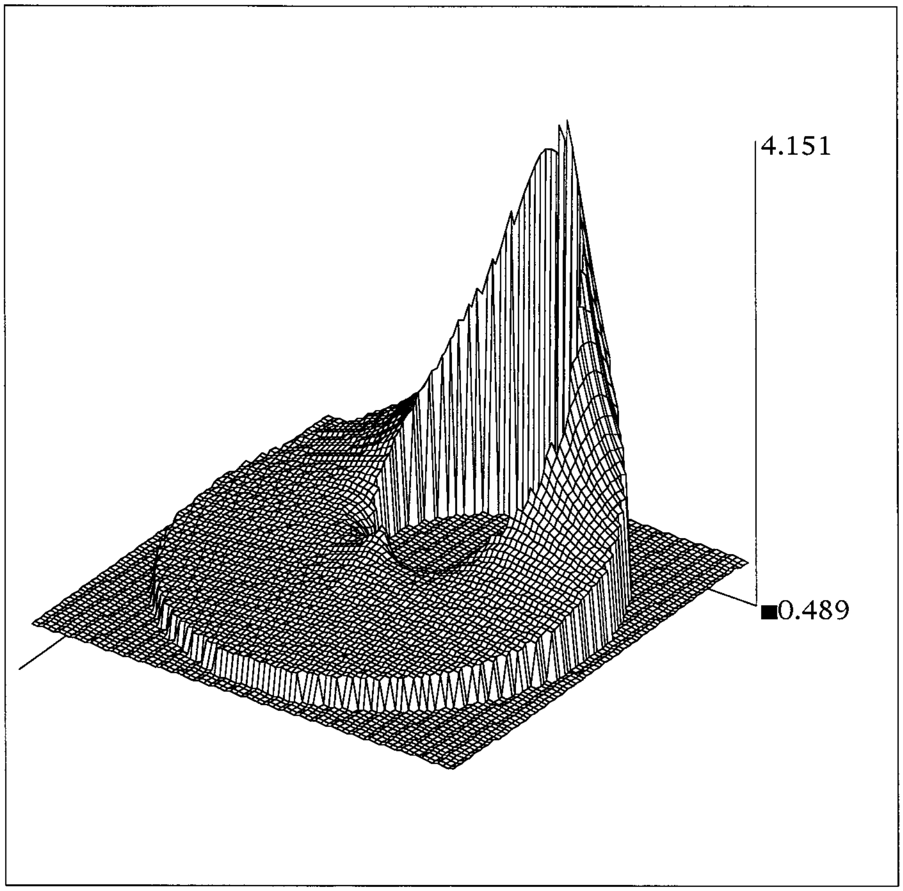

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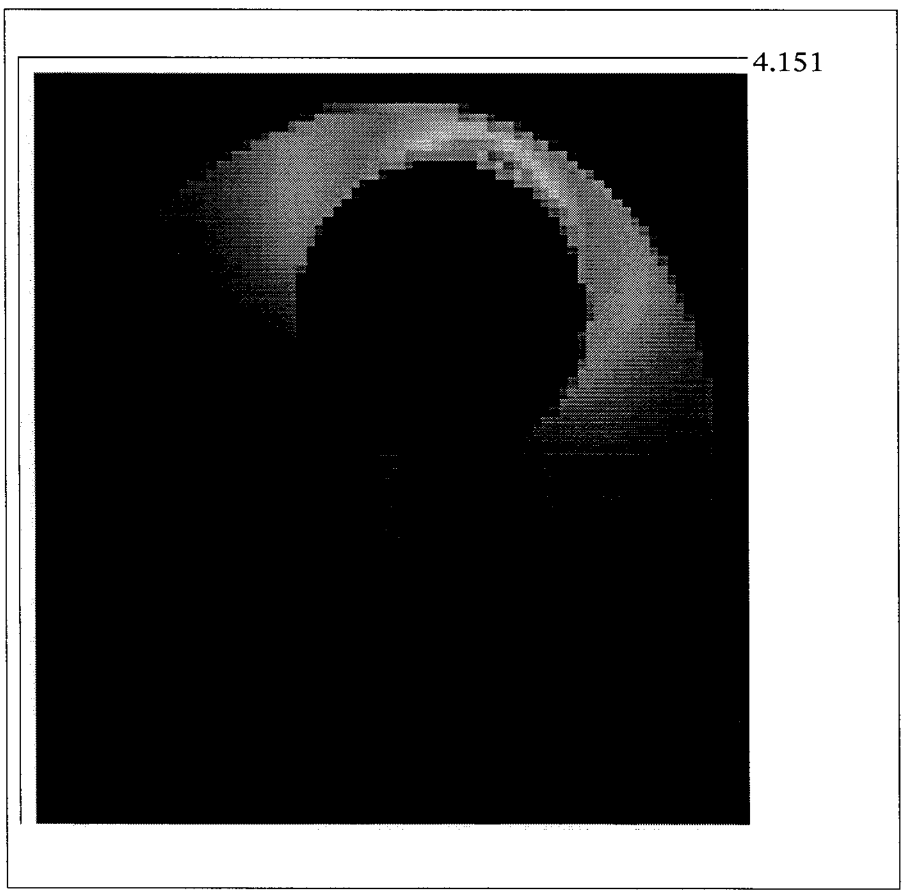

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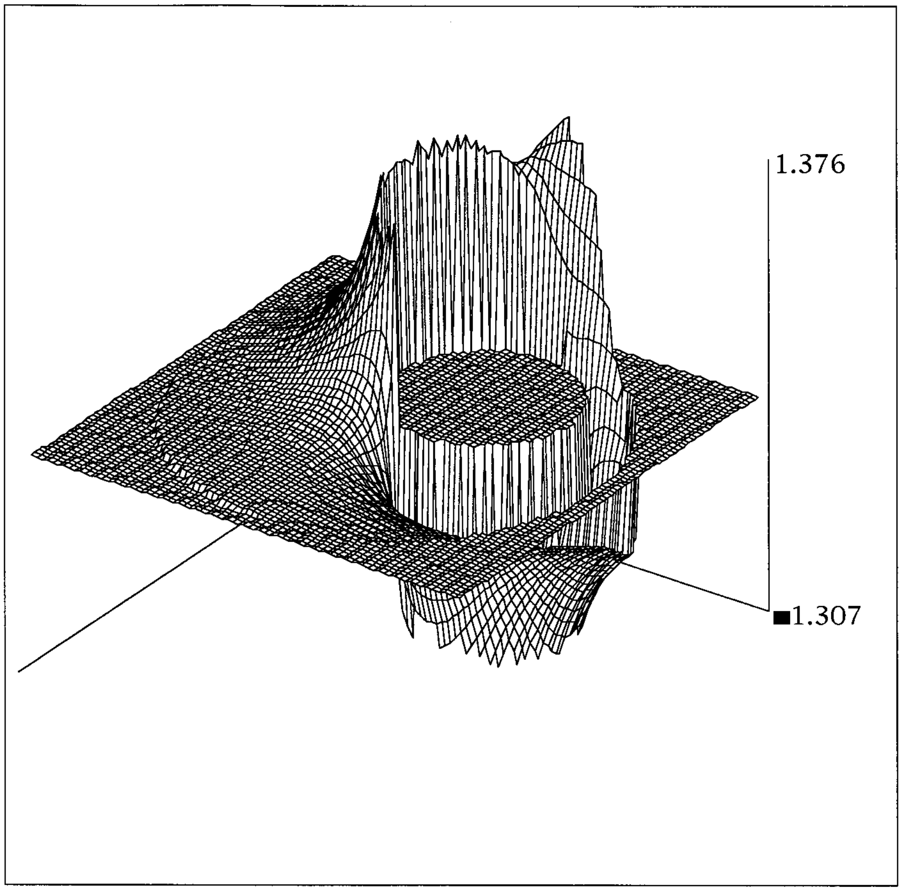
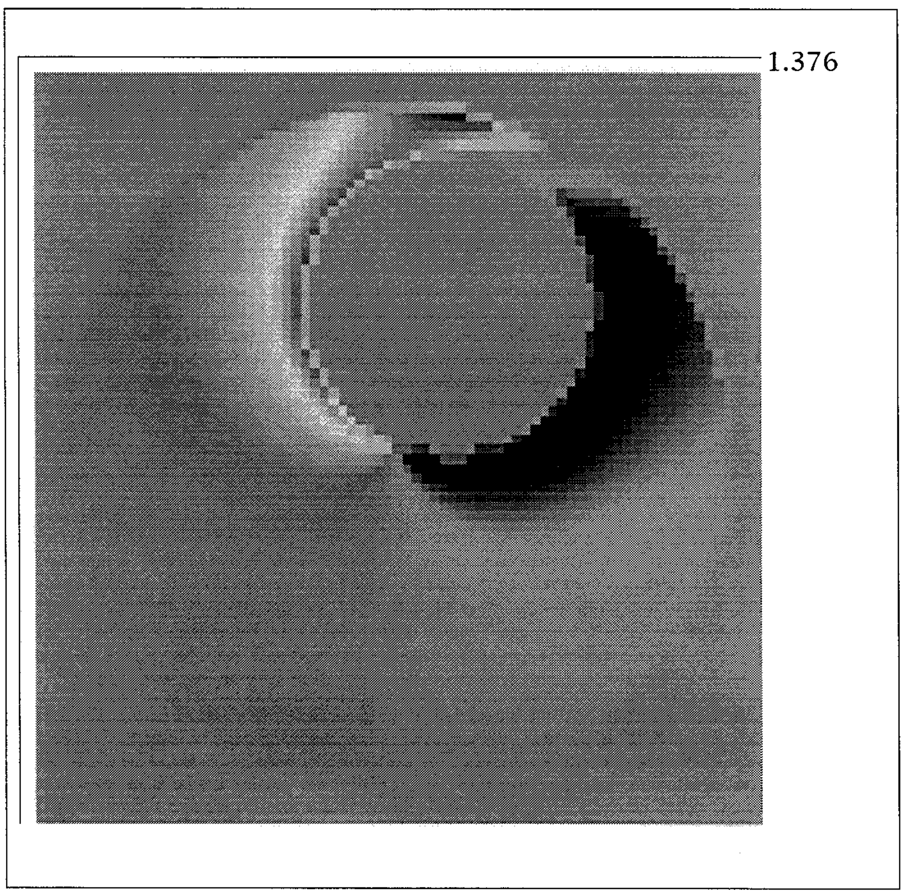
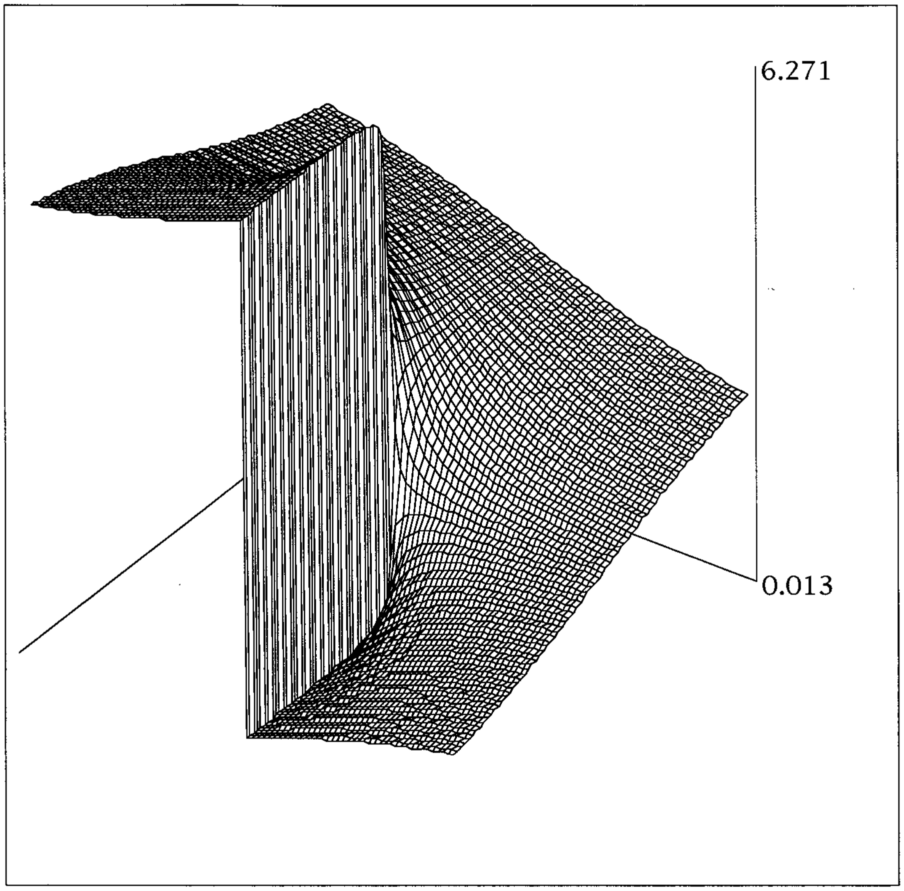


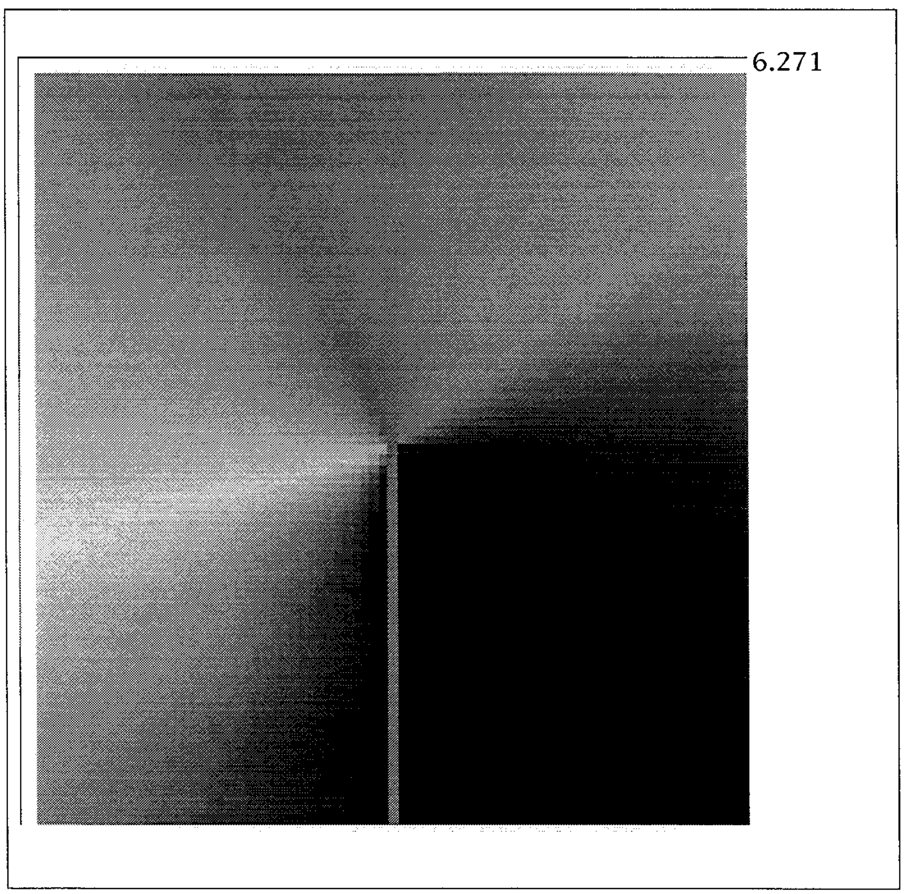
Eθ



Eθ



θ



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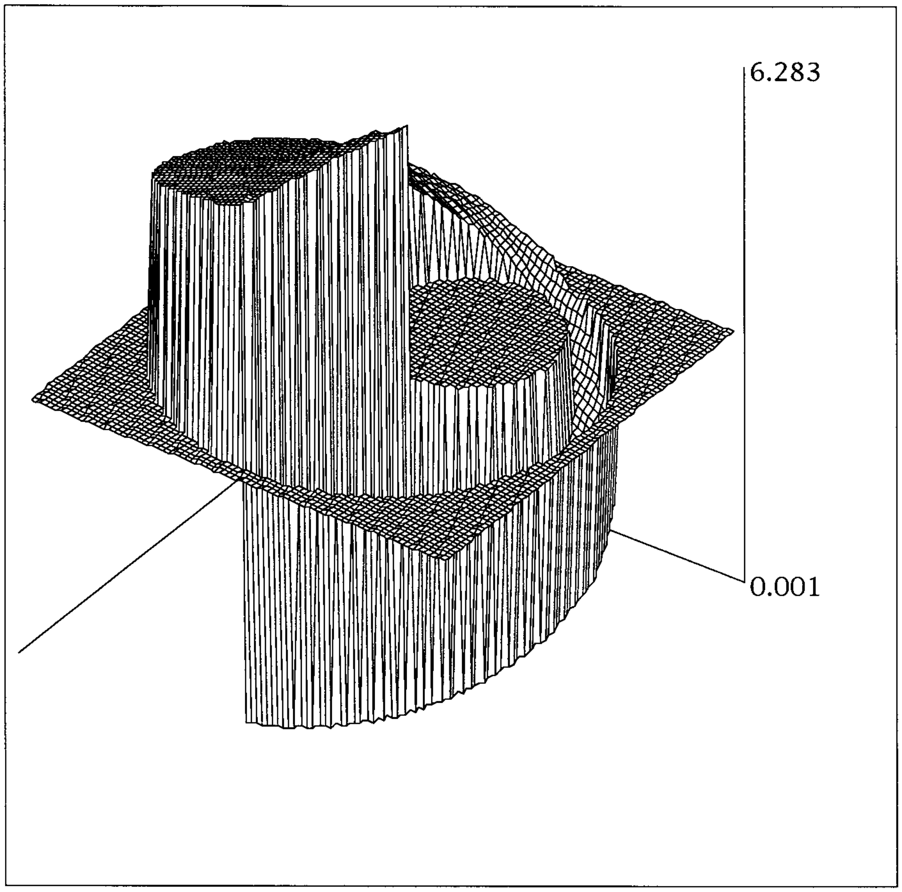
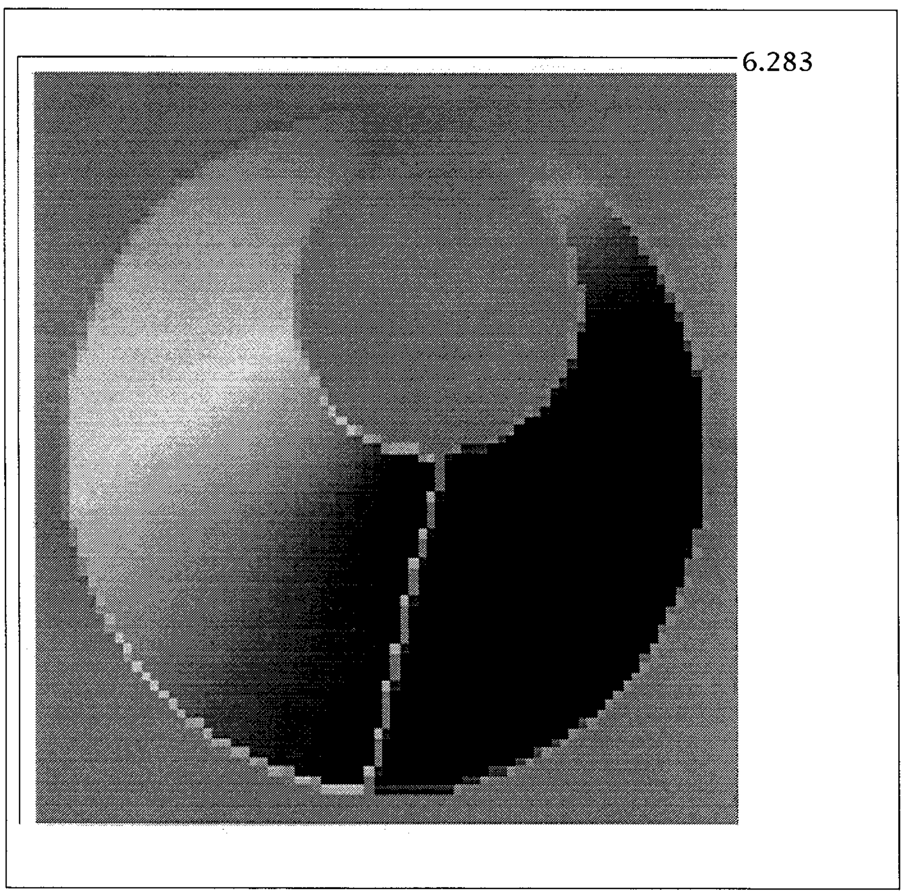


θe



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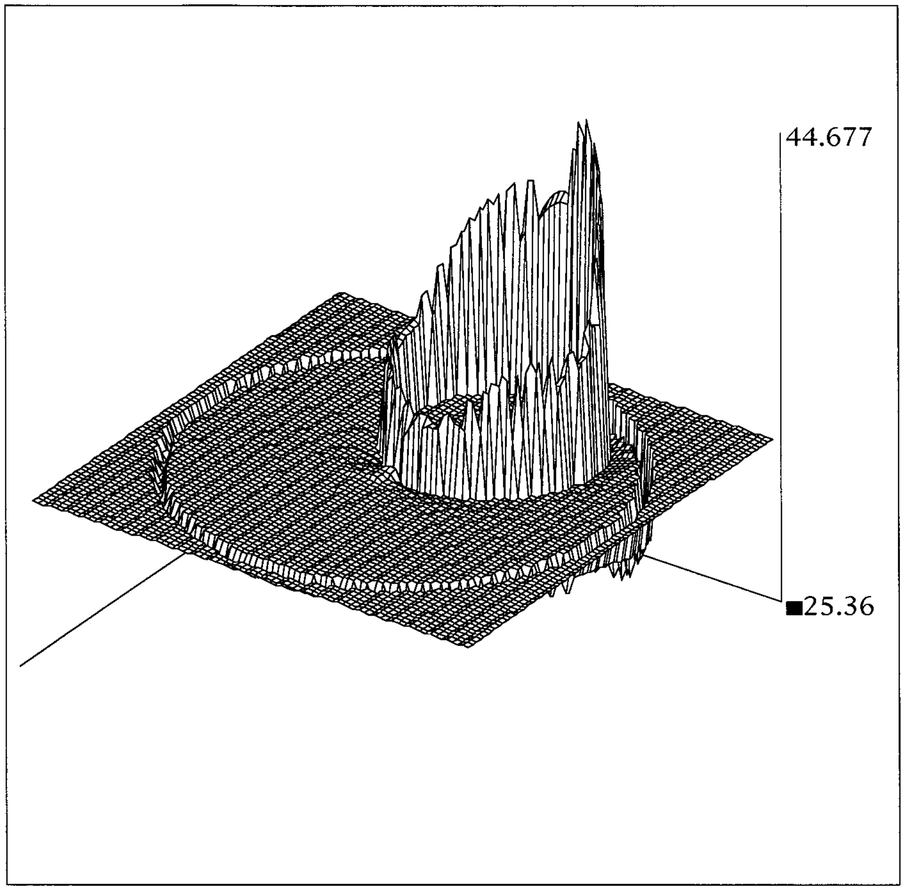
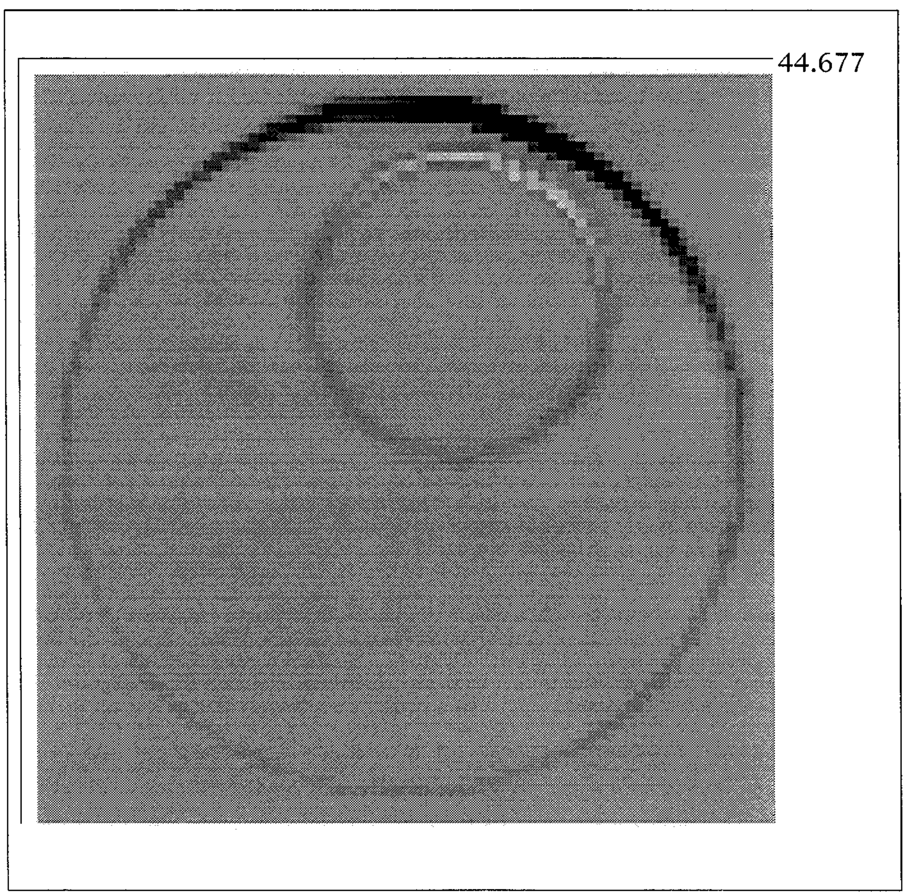


dE



dE

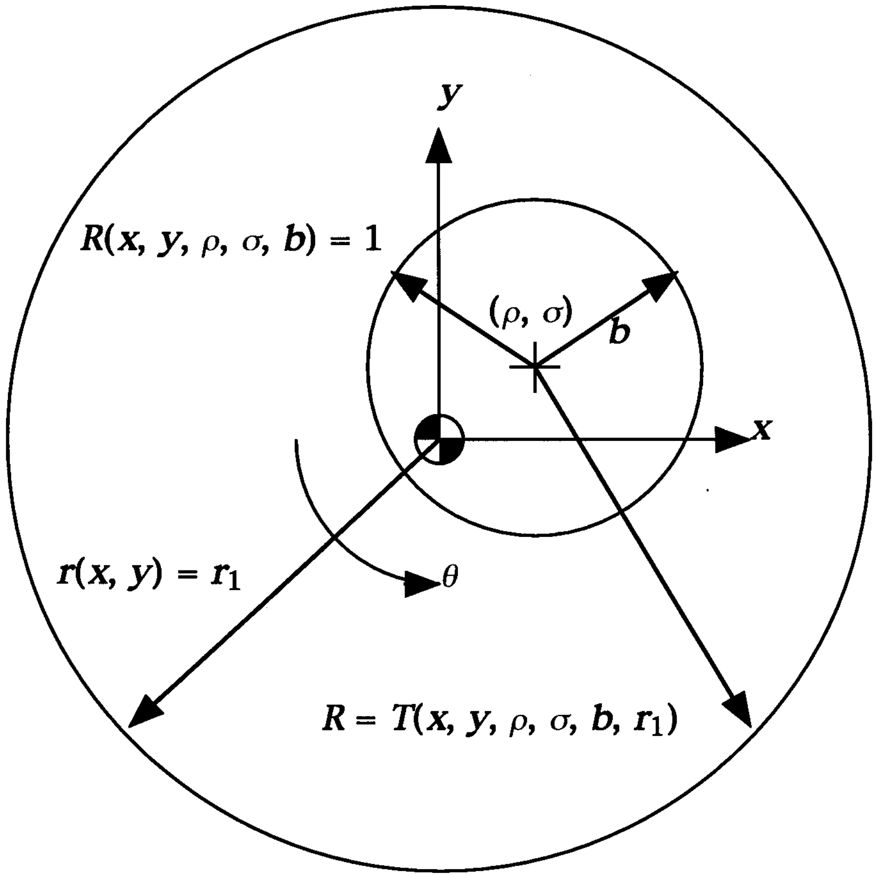


Figure 13, Fessenden's electrostatic problem
Find $V(x, y)$ between an inner cylinder at $V = 1$, and an outer cylinder at $V = 0$; also find the radial and azimuthal components of the electric field, E_r and E_θ, respectively.