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Acoustic Propagation in a Water-Filled Cylindrical Pipe

E. J. Sullivan (Consultant), J. V. Candy

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I. BACKGROUND

Acoustic propagation in a cylindrical duct is governed by that solution of the wave equation satisfying the relevant boundary conditions. For the case of a water-filled cylinder of radius r_p and a radian frequency $\omega = 2\pi f$, the solution for the acoustic pressure due to a point source at $r = r_0$, $\phi = \phi_0$ and $z = z_0$ is given by

$$p(r,\phi,z) = \sum_{m=1}^{M} [J_m(\gamma r_0)\Phi_m(\phi_0)]J_m(\gamma r)\Phi_m(\phi)e^{i\kappa_m(z-z_0)} . \tag{1}$$

Here, $k = 2\pi f / c_w$ and c_w is the speed of sound in water. Referring to Figure 1, the coordinate system is such that *z* is along the axis of the pipe; *r* is the radial coordinate inside the pipe and ϕ is the angle about the *z*-axis. The solution is found by the method of separation of variables, and the derivation is given in Appendix A.

The radial and longitudinal wavenumbers must obey the *dispersion* equation,

$$k^2 = \gamma^2 + \kappa^2 . \tag{2}$$

The $J_m(\gamma r)$ term is the Bessel function of the first kind of order *m* and $\Phi_m(\phi)$ can be either $\sin(m\phi)$, $\cos(m\phi)$ or a combination of the two, depending on the choice of ϕ_0 .

The final step is to select from the general solution for $p(r,\phi,z)$ only those solutions that satisfy the boundary conditions at the wall of the pipe. Here, we shall assume that the pipe is infinitely hard, which imposes the condition

$$J'_m(\gamma r) = \frac{\partial}{\partial r} J_m(\gamma r)\Big|_{r=r_p} = 0 ,$$

where r_p is the radius of the pipe. This boundary condition is valid for the case of a water-filled steel pipe at high frequencies, such as we deal with here. At lower frequencies, *i.e*., near the ring frequency of the pipe, the waves in the wall of the pipe must be considered [1]. A MATLAB code was written to locate the zeros of J'_m. In this computation, there are actually two indices of concern, *i.e.,* the index (order) of the Bessel function, and *l,* the index of the arguments where the zeros occur. For each *m,* there is generally more than one value of *l.* Also, since we are only interested in propagating modes, we must consider only those values of γ_{ml} that satisfy

$$\kappa_{ml} = \sqrt{k^2 - \gamma^2_{ml}} \tag{3}$$

for κ_{ml} real. Thus propagation will occur only for values of γ_{ml} that satisfy $\gamma_{ml} \le k$.

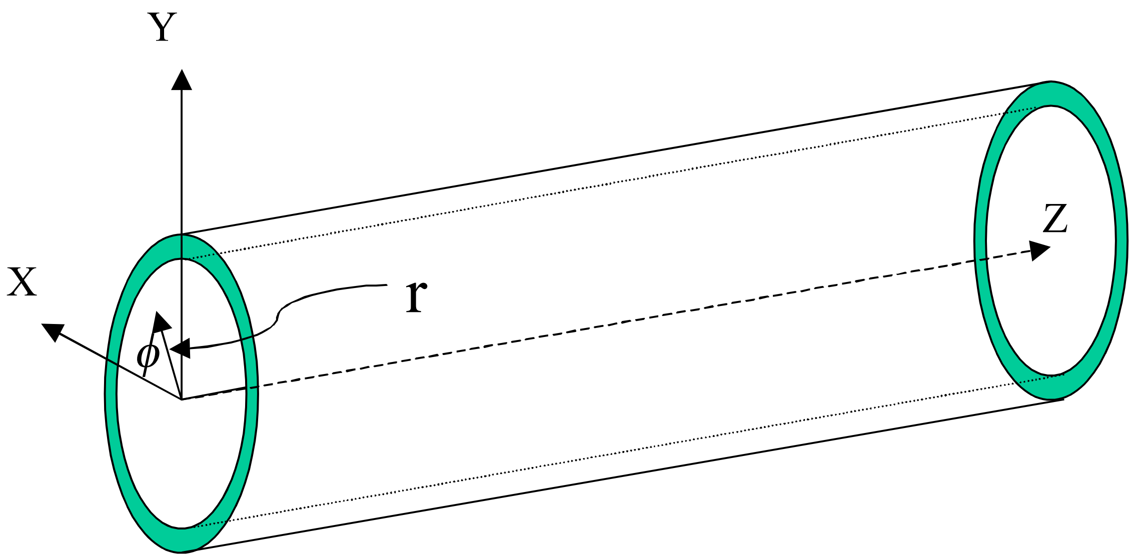

Figure 1. Cylindrical coordinate system. The z axis is on the centerline of the pipe and is the direction of propagation.

As an example, if we choose a frequency of *f*=37.5 kHz and c_w=1480 m/sec, the largest allowable value of γ_{ml} is found to be

$$\max \gamma_{ml} = k = \frac{2\pi(37.5)}{1.48} = 159.2 m^{-1} . \tag{4}$$

For a radius of $r_p = .2$ meters, all values of $\gamma_{ml} r_p$ satisfying $\gamma_{ml} r_p < 159.2 \times .2 = 31.8$ and satisfying the boundary condition will support propagation. This results in 5995 modes. This is a huge number, and will be the cause of some concern, as will be seen below.

The 0,0 mode is, in fact, a plane wave with speed c_w, the speed of sound in water. This is due to the fact that the first zero of $J_0^{'}(x)$ is at *x=0,* and from the dispersion equation, we see that this leads to $\kappa_{00} = k$. This is the only mode that will be supported for frequencies below the cutoff frequency of the 1,0 mode, which is found to be 2.17 kHz. A thorough discussion of this phenomenon is found in Reference 2.

As mentioned above, there are two indices specifying the eigenfunctions. One is the *order* of the Bessel function and the other specifies the *zero* of the normal (to the wall) derivative of the Bessel function. This differs from, say, the solution for a rectangular waveguide, where there is only one kind (order) of eigenfunction, *i.e.,* a cosine function in the case of an infinitely hard wall. Here, there would only be one index, that specifying the relevant zeroes. Thus, in the case of a cylindrical waveguide, there are many more eigensolutions as compared to a similar rectangular waveguide.

II. BROADBAND SOLUTION

In the case of pulse propagation the broadband solution must be considered. This is done by transforming the time domain representation of the pulse into the frequency domain, finding the solution for each frequency and recombining these solutions via an inverse Fourier transform.

Suppose we have a pulse given by the time domain expression $f(t)$. Let the discrete complex frequency domain representation of the pulse be given by F_k, where k is the frequency index. The sample rate is chosen to exceed the Nyquist criterion. Usually, $8f_{\max}$ is chosen [3]. Given the discrete complex spectrum, the time domain solution is found from the discrete inverse Fourier transform, viz.,

$$p(t_n) = \sum_{k=1}^{K} F_k e^{i\omega_k (t_n + T)} \quad . \tag{5}$$

Here, T is the travel time required for the fastest mode to reach the axial point z, so that all of the modes will be included in the time domain representation, and F_k is given by the solutions of Equation 1 for the frequencies ω_k and distance z. Equation 5 then, represents the solution for time beginning with the arrival of the fastest mode. This is sometimes referred to as the *reduced* time.

III. DISPERSION

The fact that all modes do not travel at the same speed results in what is called "dispersion." This means that the input pulse is "pulled apart" as it travels along the pipe, so that a pulse of length t_0 must, of necessity, have a final length of $t_f > t_0$. The value of t_f can be estimated by consideration of the so-called *group velocity* of the fastest and slowest modes. In our case, the fastest mode is the 0,0 mode, which is a plane wave and has a group velocity equal to its phase velocity, which is c_w, the speed of sound in water. All higher modes will have group velocities less than this. This can be somewhat counter intuitive, since all of the higher modes have *phase velocities* greater than c_w. However, the acoustic *energy* cannot travel faster than c_w. From [3] the group velocity for a mode propagating with axial wavenumber κ_m, is given by

$$u_{g,m} = \left(\frac{\kappa_m}{k}\right) c_w \quad . \tag{6}$$

IV. EXAMPLE

We now consider the propagation of a pulse with center frequency of 37.5 kHz defined by

$$f(t) = .5(1 - .75\cos .0625\omega_c t)\sin \omega_c t \quad \textit{for} \quad 0 < t < 4/f_c \quad \text{and zero elsewhere.} \tag{7}$$

This pulse is depicted in Figure 2. A full time window of .004 seconds was used, with the pulse taking up the first 10% of this window. With a total of 2048 samples over the time of .004 seconds, this leaves 210 samples for the pulse itself, and a sample rate of 512 kHz. The spectrum of this pulse is shown in Figure 3, where it is seen that it has a bandwidth on the order of 5 kHz.

We are now in a position to estimate the pulse spread due to dispersion. For our example, we choose a steel pipe of radius .2 meters and investigate the configuration of the pulse at the point z=1000 meters.

As mentioned before, the fastest mode arrives at the speed of sound in water, which is 1480 m/s. The slowest mode containing any significant energy is found to have a group velocity of about 1175 m/s. Thus we see that at a distance of 1000 meters, our pulse of length .0004 sec, has spread to a length of

$$t_f = 1000(1175^{-1} - 1480^{-1}) = .175\,\mathrm{sec}\,.$$

To see more clearly what is taking place, A MATLAB code was written which computes the field in the pipe at a given distance *z*. In Figure 4 we see the plane wave pulse, *i.e.,* the 0,0 mode at z=1000 meters. If the highest frequency contained in the pulse were less than the cutoff frequency of the next higher mode, which is the 0,1 mode and has a cutoff frequency of 2.17 kHz, then this is all that would appear at any distance along the pipe. However, the frequencies of concern are much higher than this. In Figure 5, the 0,0 and 0,1 modes are shown at z=1000 meters, where we see that the 0,1 mode is significantly slower than the 0,0 mode. Continuing, Figure 6 shows the sum of the modes up to the 0,2 case, Figure 7 shows the sum of the modes up to the 1,2 case, and Figure 8 shows the sum of the modes up to the 2,2 case. Finally, we show the case of m=1 with *all* of the allowed zeroes, which for our case, is *l*=69. Since the allowed values of m is 213 and the total number of modes is 5995, we see that the situation is extremely complex, and that in general, any pulse with such high frequencies will be greatly distorted by dispersion.

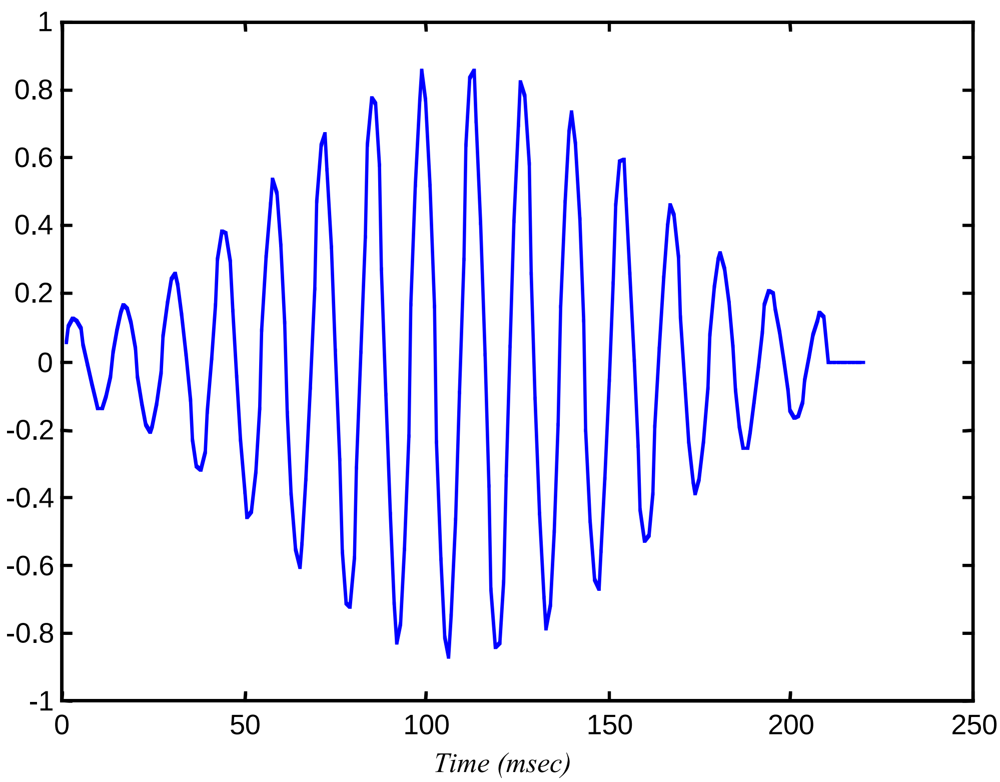

Figure 2. **Test pulse in the time domain. The pulse is made up of 210 samples, which represents .0004 seconds.**

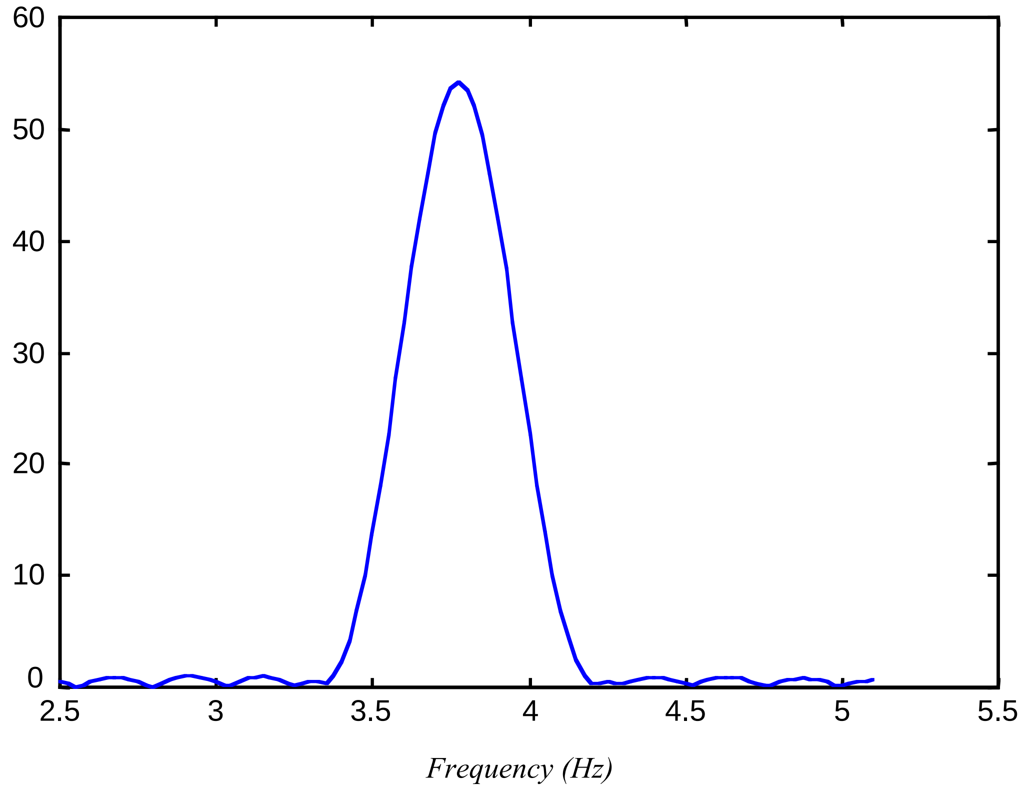

Figure 3. **Magnitude spectrum of the pulse shown in Figure 2.**

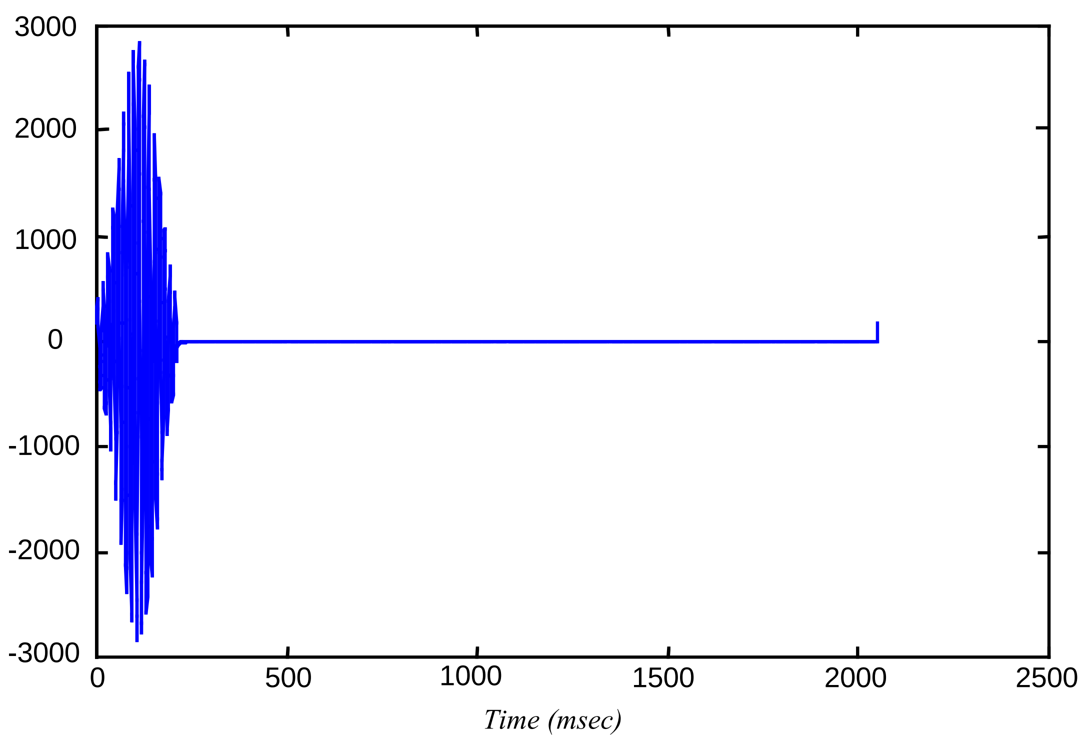

Figure 4. **Time domain representation of the first .004 seconds (2048 samples) of the field at z=1000 meters for the 0,0 case.**

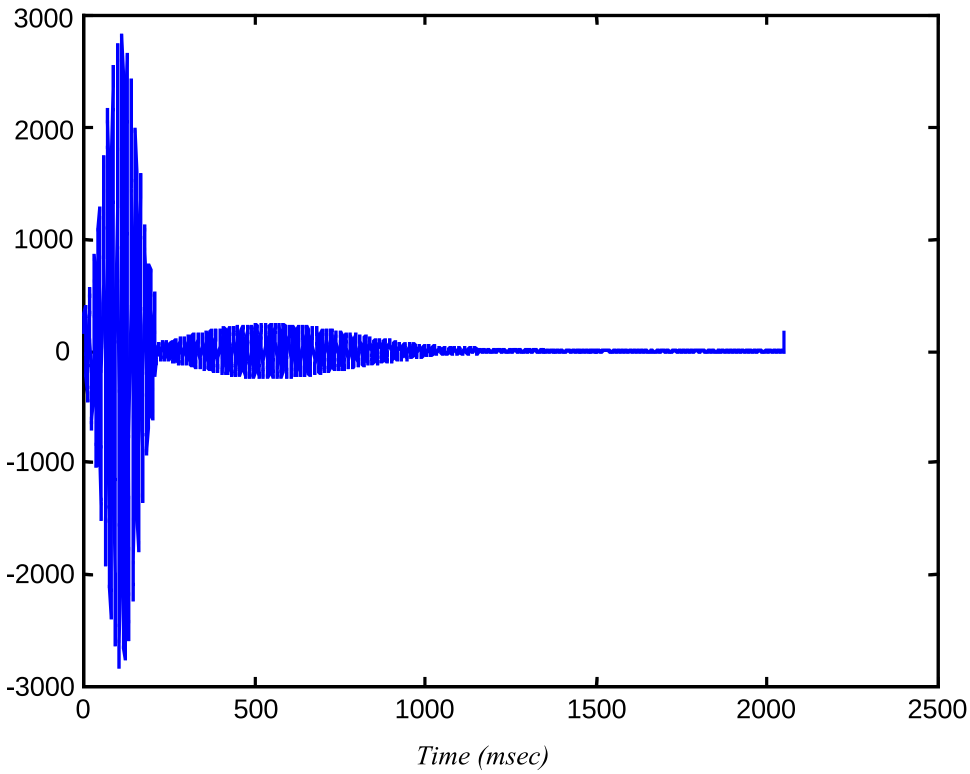

Figure 5. Time domain representation of the first .004 seconds (2048 samples) of the field at z=1000 meters for the 0,1 case.

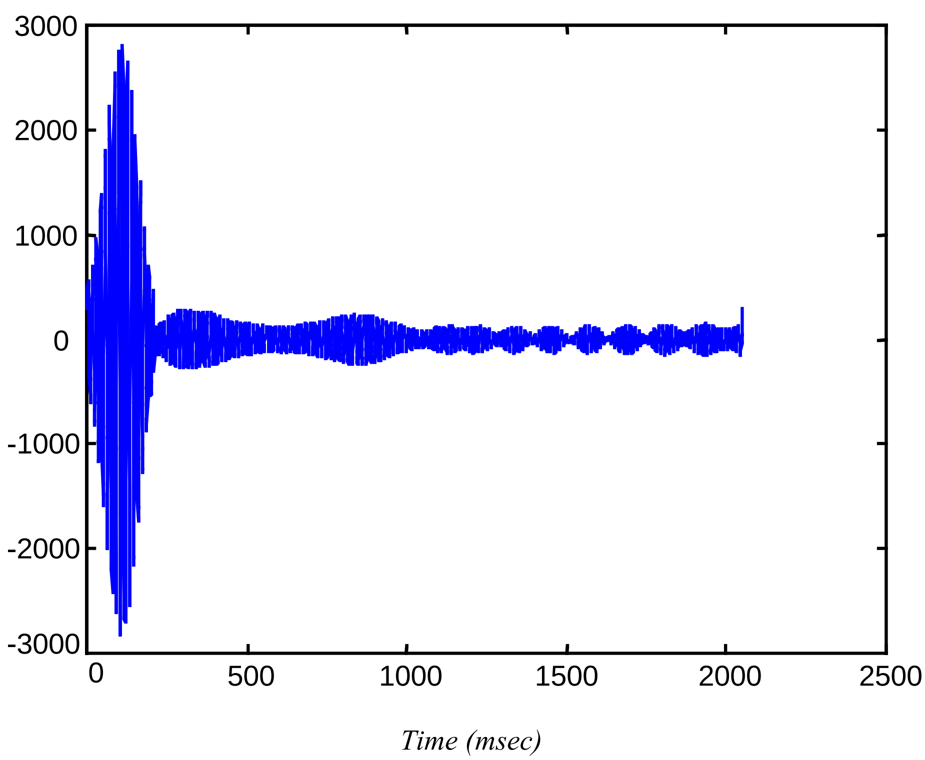

Figure 6. Time domain representation of the first .004 seconds (2048 samples) of the field at z=1000 meters for the 0,2 case.

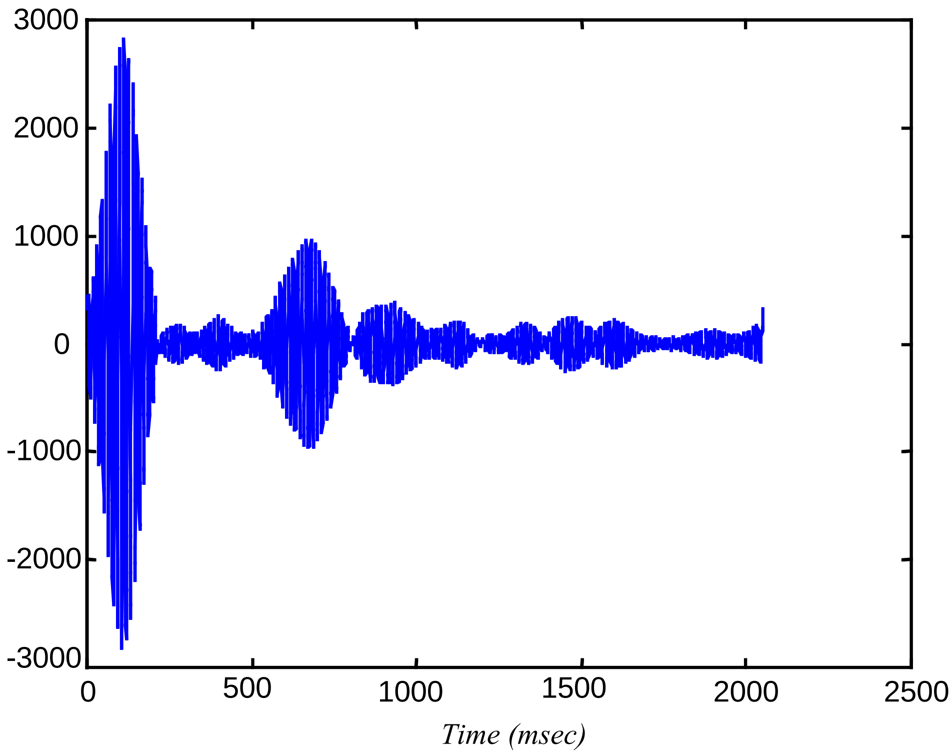

Figure 7. **Time domain representation of the first .004 seconds (2048 samples) of the field at z=1000 meters for the 1,2 case.**

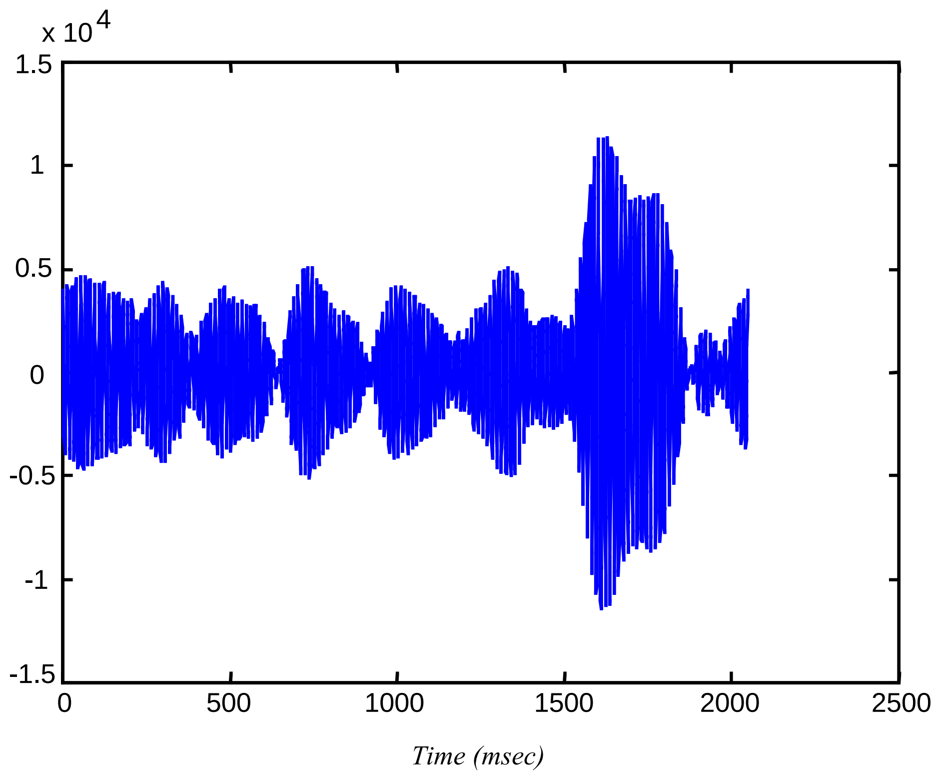

Figure 8. **Time domain representation of the first .004 seconds (2048 samples) of the field at z=1000 meters for the 2,2 case.**

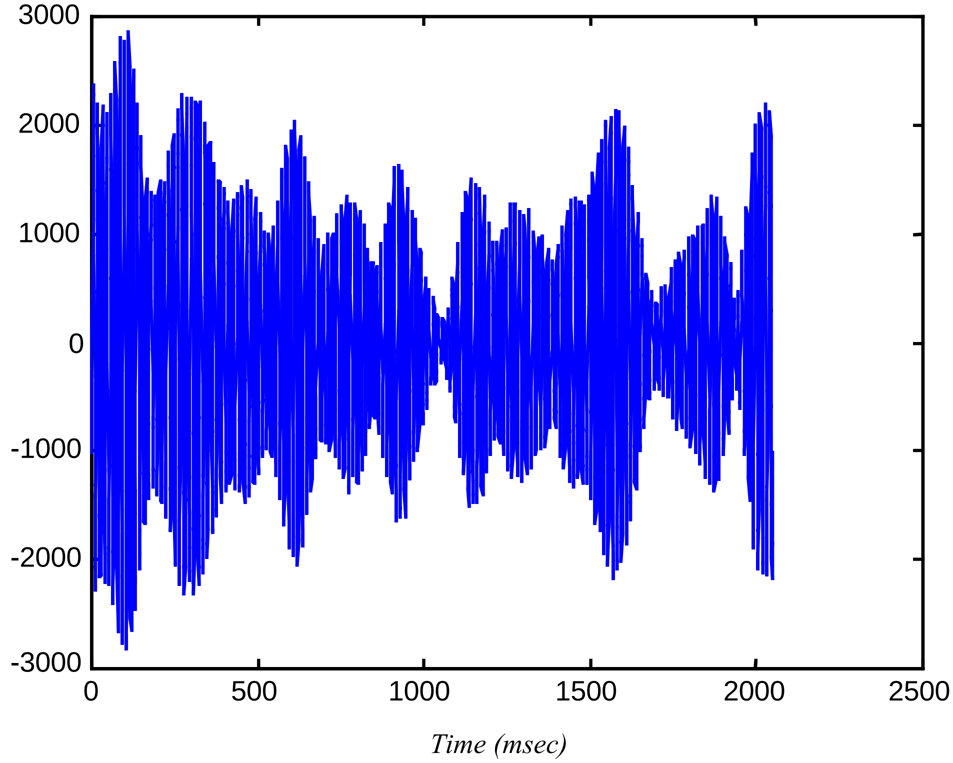

Figure 9. Time domain representation of the first .004 seconds (2048 samples) of the field at z=1000 meters for the 1,69 case.

V. SIGNAL PROCESSING ISSUES

If it is desired to transmit information by using tone bursts such as the one discussed above, there are two major issues of concern. The first is that the early arrival times should be examined to extract the 0,0 mode from the received pulse. We assume here that this could be done with a matched filter. The second issue is that most of the energy of the pulse has been distributed over the many modes, thereby considerably weakening the 0,0 mode, which is the mode of interest.

VI. COMMENTS

This study was concerned with the physics of the propagation of a tone burst of high frequency sound in a steel water-filled pipe. The choice of the pulse was rather arbitrary, so that this work in no way can be considered as recommending a particular pulse form. However, the MATLAB computer codes developed in this study are general enough to carry out studies of pulses of various forms. Also, it should be pointed out that the codes as written are quite time consuming. A computation of the complete field, including all 5995 modes, requires several hours on a desktop computer. The time required by such computations as these is a direct consequence of the bandwidths, frequencies and sample rates employed. No attempt was made to optimize these codes, and it is assumed that much can be done in this regard.

REFERENCES

1. L. D. Lafleur and F. D. Shields, "Low-frequency propagation modes in a liquid-filled elastic tube waveguide," *J. Acoust Soc. Am.,* **97**, (3), March 1995 pp 1435-1445.

2. P. M. Morse and K. Uno Ingard, *Theoretical Acoustics,* McGraw-Hill Book Co., New York, 1968 pp 509-513.

3. F. B. Jensen, et al., *Computational Ocean Acoustics,* AIP Press, New York, 1994 Chap. 8.

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APPENDIX A

Appendix A

DERIVATION OF THE FIELD EQUATION

The propagation of acoustic waves in a fluid-filled cylindrical pipe, for a single frequency *f*, is governed by the Hemholz equation. In cylindrical coordinates, the Hemholz equation for a point source located at r_0, ϕ_0 and z_0, is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\left(\frac{1}{r^2}\right)\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=\frac{\delta(r-r_0)\delta(\phi-\phi_0)\delta(z-z_0)}{2\pi r} \tag{A-1}$$

Here, $k=2\pi f/c$ and *c* is the speed of sound in the fluid. As shown in Figure 1, the coordinate system is such that *z* is along the axis of the pipe, *r* is the radial coordinate inside the pipe and ϕ is the angle about the *z* axis. The solution is found as follows. First the homogenous equation is solved by the method of separation of variables. Here, the solution is assumed to be a product of three functions, as follows.

$$\psi(r,\phi,z)=R(r)\Phi(\phi)Z(z)\,. \tag{A-2}$$

Substituting into the *homogenous* equation, *i.e.,* Equation A-1 with the source term set to zero, yields

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+k^2=0\,. \tag{A-3}$$

The form of this equation requires that the third term on the L.H.S. be a constant, *i.e.,*

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-\kappa^2\,, \tag{A-4}$$

so that Equation A-3 becomes

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}+k^2-\kappa^2=0\,. \tag{A-5}$$

Since κ^2 and k^2 are both constants, it follows that the first two terms on the L.H.S. of Equation A-5 must sum to a constant, it follows that

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-\gamma^2\,. \tag{A-6}$$

This leads to the so-called dispersion equation, given by

$$k^2=\gamma^2+\kappa^2\,. \tag{A-7}$$

We now multiply Equation A-6 by r^2 and conclude that

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2, \tag{A-8}$$

is also a constant. We now have the following equation for R.

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + (k_r^2 - \frac{m^2}{r^2}) = 0. \tag{A-9}$$

This is Bessel's Equation, and since there is no propagation transverse to the z axis, only Bessel Functions of the first and second kinds are relevant. Furthermore, since the Bessel Function of the second kind is singular at the origin, it does not apply to our problem. Thus only the Bessel Function of the first kind is relevant, so that we finally have, using standard notation,

$$R(r) = J_m(\gamma r)\,. \tag{A-10}$$

It quickly follows that

$$\Phi_m(\phi) = \begin{cases} \sin\phi \\ \cos\phi \end{cases} \tag{A-11}$$

and since we shall consider propagation in the $+z$ direction only,

$$Z(z) = e^{i\kappa z}\,. \tag{A-12}$$

The general homogenous solution is then given by

$$p(r,\phi,z) = \sum_{m=1}^{M} a_m J_m(\gamma r)\Phi_m(\phi)e^{i\kappa(z-z_0)}\,. \tag{A-13}$$

Substituting this into the inhomogeneous differential equation, Equation A-1, and using the orthogonality of J_m and the Φ_m, the value of a_m is found to be

$$a_m = J_m(\gamma r_0)\Phi(\phi_0)\,, \tag{A-14}$$

so that the general solution to the inhomogeneous equation is

$$p(r,\phi,z) = \sum_{m=1}^{M} \left[J_m(\gamma r_0)\Phi(\phi_0)\right] J_m(\gamma r)\Phi_m(\phi)e^{i\kappa(z-z_0)}\,. \tag{A-15}$$

Appendix B

MATLAB PROGRAM

Program BBModesum.m

```
%
%  Program BBmodesum.m
%  Computes acoustic field in cylindrical pipe for given
frequency and
%  pipe radius in meters
clear
tic
global m
d2r=pi/180;
%
% Pipe radius
rad=.2;
%
%Speed of sound
cee=1480;
%
%  Define source position
r0=.15;
thet0=0*d2r;
%
%  Define "look" coordinates of field in transverse (r, theta)
space
frac=.7;
r=frac*rad;
ang=15;
thet=ang*d2r;
%
%  Find signal spectrum
%
%LLsig generates signal time series and spectrum
LLsig
%
% flines is signal spectrum
flines;
M=length(flines);
%
%  Loop on frequency
for k=1:M
   ff=freq(k);
%
% Find zeroes of Bessel function derivatives
   [alf,kay] = wavnum(rad,ff,cee);
blf=alf;
blf(1,1)=1;
Lm=length(blf(:,1));
for im=1:Lm
```

```
   Lz(im)=length(blf(find(blf(im,:))));
end
%
%  Generate radial wavenumbers
gam=alf/rad;
%
%  Generate axial wavenumbers
kap=sqrt(kay.^2-gam.^2);
%
%  Generate coefficients
ep=2;
for m=1:Lm
   if m==1
      ep=1;
   else
      ep=2;
   end
   for n=1:Lz(m)
      den(m,n)=alf(m,n);
      den(1,1)=1;
      normm(m,n)=pi*rad^2*(1-((m-1)/den(m,n))^2)*besselj(m-
1,alf(m,n))^2/ep;
      A(m,n)=besselj(m-1,gam(m,n)*r0)*cos(m-1*thet0);
      coeff(m,n)=A(m,n)/normm(m,n);
   end
end
%
%  Generate field sum
z=1000;
PF(k)=0;
for m=1:Lm
%for m=2:Lm
for n=1:Lz(m)
   %if n>3,break,end
      PF(k)=PF(k)+coeff(m,n)*besselj(m-1,gam(m,n)*r)*cos(m-
1*thet)*exp(i*kap(m,n)*z);
   end
end
%ugrp=(kap/kay)*cee;
end
%umin=min(ugrp)
%
%  Compute output time series
%
%  Fastest wave is zeroth mode or group velocity=cee. Thus
tstart is cee/z.
%
tstart=(z/cee);
%
%  Compute IFT of PF(k)
tobs=[tstart:dt:tstart+T];
```

```
arg=i*2*pi*freq';
expy=exp(arg*tobs);
tout1=conj(PF).*(flines);
tout=tout1*expy;
plot(real(tout))
toc
```

Program LLsig.m

```
%
%  Program LLsig
%
% Define time series
%
% fc is center frequency of pulse
fc=37500;
omegc=2*pi*fc;
%
% T is time window size
T=.004;
nsamps=2048;
fsamp=nsamps/T;
dt=T/nsamps;
tser=[dt:dt:T];
%
% Ints is input signal
Ints=.5*sin(omegc*tser).*(1-.75*cos(.0625*omegc*tser));
Ints(210:2048)=0;
%  Compute spectrum
specc=fft(Ints);
NN=length(specc);
fax=fsamp*[1:(NN)]/NN;
flines=specc(1:(NN)/2);
freq=fax(1:(NN)/2);
```

Program wavnum.m

```
function[alf,kay] = wavnum(rad,freq,cee);
%
% This finds the zeroes of the derivatives of Bessel functions
of the first kind
% for the case of a cylinder with radius "rad" for frequency
"freq" for all
% propagating modes. "cee" is the speed of sound. Nm is the
% number of propagating modes and kay is the acoustic
wavenumber.
% Uses dJm/dx=m*besselj(m,x)-x*besselj(m+1,x) from P. M. Morse's
"Vibration and Sound,"
% McGraw-Hill, 1948, p-188.
global m
kk=0;
kay=2*pi*freq/cee;
maxarg=kay*rad;
del=maxarg/100;
maxy=1000000;
%
% Loop on order of Bessel function
%
alf(1,1)=1;
for mm=1:maxy
   m=mm-1;
%
% Find appx. zero crossing by testing for sign change
%
arg=0;
j=1;
sn(1)=sign(m*besselj(m,del)-del*besselj(m+1,del));
for i=2:101
   arg=arg+del;
   sn(i)=m*besselj(m,arg)-arg*besselj(m+1,arg);
   tst=sn(i)*sn(i-1);
   if tst<0
      argz(j)=arg;
      j=j+1;
   end
end
%
% Find exact zeroes
%
for k=1:j-1
   x=argz(k);
    if mm==1
      k=k+1;
    end
   kk=mm;
   alf(mm,k)=fzero('bessyprime',x);
```

```
end
 if mm>kk,break,end
 end
alf(1,1)=0;
 %
 % Order by size
 %
 %alf1=find(alf);
 %alf2=alf(alf1);
 %Len=length(alf2);
 %Nm=Len+1
 %alfor(1)=0;
 %alfor(2:Nm)=sort(alf2)';
```

Program bessyprime

```
% Program bessyprime
%
% Makes function of Bessel function derivative for use in fzero.
% Uses dJm/dx=m*besselj(m,x)-x*besselj(m+1,x) from P. M. Morse's
"Vibration and Sound,"
% McGraw-Hill, 1948, p-188.
%
function y=bessyprime(x,m)
global m
y=m*besselj(m,x)-x*besselj(m+1,x);
```

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