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Stretched and Filtered Transport Synthetic Acceleration of S_N Problems. Part 2: Heterogeneous Media

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INTRODUCTION

In [1], we presented the stretched filtered transport synthetic acceleration method (SFTSA) for homogeneous media. Both SFTSA and SFTSA preconditioned Krylov were shown to be effective iterative schemes in homogeneous media due to the predictable structure of the iteration eigenvalues. In heterogeneous media or on non-uniform grids, the eigenvalue structure is unpredictable for general problems, making the filter strength α for optimal SFTSA extremely problem dependent. Leaving α set to the optimal value (in each cell by table lookup) predicted by homogeneous media theory can make SFTSA divergent, even for relatively mild heterogeneities. Thus, SFTSA is more fragile than DSA in the sense that most DSA schemes break down only for much more severe heterogeneities. Fortunately, breakdown of SFTSA occurs with large negative eigenvalues, and Krylov methods preconditioned with SFTSA remain effective for such problems. Therefore, with a Krylov scheme "wrapped around" SFTSA, the resulting method is relatively insensitive to the filter strength, and a user may achieve reasonably good performance, if not optimal, with a fixed α over a wide range of heterogeneous problems.

HETEROGENEOUS SFTSA

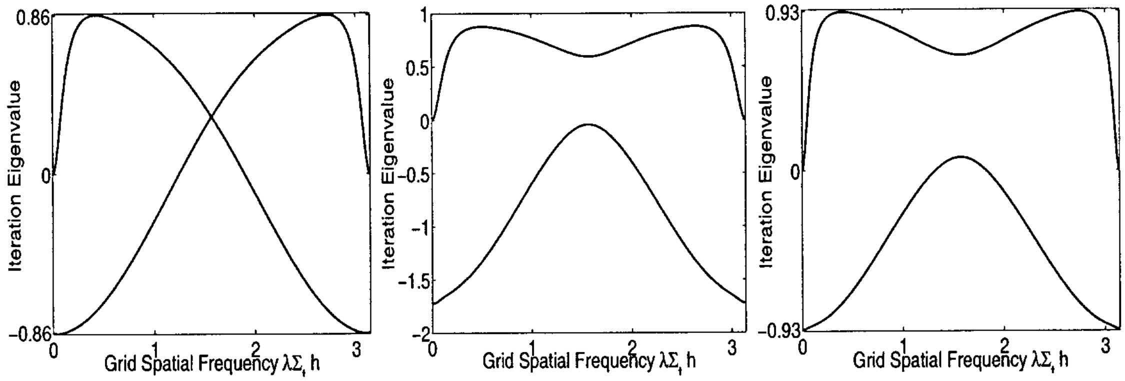
To stretch the exact error transport equation into a single pure absorber problem, the stretch ϵ in each cell depends on the material properties of that cell. Cells with different scattering ratios have different stretches, such that the mean free path in each cell of the stretched problem is $\sqrt{3}L$ for the cell, where L is the diffusion length. The stretch in a heterogeneous problem is not uniform across the grid; hence, there is not a P_1 equivalence between the stretched transport problem and the associated stretched diffusion approximation. This is an underlying physical reason for some of the instabilities related to heterogeneities. This also means that the boundary condition for the stretched problem corresponding to an inhomogeneous boundary condition in the original problem is allowed to be a vacuum boundary instead of the albedo desribed in Part I. (In practice, this prevents generation of spurious near-singular eigenvalues that may cause restarted GMRES to stagnate.)

Unfortunately, for c close to unity, relatively mild heterogeneities can have a large negative impact on SFTSA. In Figure 1, we show three plots of iteration eigenvalues for the two-region periodic problem indicated. Figure 1a shows a uniform medium and grid result with a spectral radius of 0.86 when the optimum filter strength α is used. The mild grid heterogeneity in Figure 1b causes the spectral radius to jump to 1.72, with the filter strength α for each cell set to optimize an infinite homogeneous problem. Increasing the filter strengths (ad hoc) as in Figure 1c removes the instability but drives the spectral radius to 0.93, closer to the source iteration value of 0.9999. With a sufficiently large filter strength, any heterogeneous problem may be stabilized, but the optimum filter strength then becomes highly problem-dependent and performance is not guaranteed to be significantly better than source iteration.

We have observed that material heterogeneities and grid heterogeneities have similar effects. For c far from unity in all cells, stronger heterogeneities are required to cause instability than in problems with c closer to unity, but a stronger filter is still required than for a homogeneous problem. From our analysis and testing in 1-D with the ES method, we conclude that SFTSA without Krylov is not viable for heterogeneous problems, except for smooth material variation on smooth grids.

HETEROGENEOUS KRYLOV SFTSA

When a Krylov solver, such as restarted GMRES, is "wrapped around" SFTSA, the method becomes much more effective. Furthermore, performance is relatively insensitive to variation in the filter strength α. Certainly, performance on a given problem can be optimized by trial and error (or by Fourier analysis for a nearly periodic heterogeneity), but performance does not significantly deteriorate if one simply fixes α for many problem types. The reason is that heterogeneities generate



(a) $\Sigma_t h = [1, 1]$,
$\alpha = 0.0854$

(b) $\Sigma_t h = [2, 0.5]$,
$\alpha = [0.1470, 0.0427]$

(c) $\Sigma_t h = [2, 0.5]$,
$\alpha = [0.19, 0.1]$

Figure 1: Periodic Two-Region SFTSA Eigenvalues for $h \equiv 1$ and $c \equiv 0.9999$.

large negative eigenvalues in SFTSA. When one "wraps" a Krylov scheme around the method and solve the linear system directly as in [2, 3], the linear system eigenvalues are unity minus the eigenvalues of SFTSA. Hence, eigenvalues of SFTSA with negative real part are pushed out to the right of unity in the linear system solved by Krylov iteration. While this can slow convergence, stagnation or divergence does not occur. When c is close to unity in many parts of the problem, the diffusive eigenvalues get pushed by the SFTSA preconditioner from close to the origin to close to unity, and this gives a tremendous speedup to the Krylov iterations.

NUMERICAL RESULTS

We consider three heterogeneous test problems in Table 1. In each case we consider a 10,000 cell planar grid with vacuum boundaries, a uniform volume source $Q = 0.01$, and an S_{16} quadrature set. We compare SFTSA (labelled SFT) with $\alpha \equiv 0.3$ to (i) unpreconditioned GMRES (restarted with a maximum subspace dimension of 20 and labelled G), (ii) SFTSA preconditioned GMRES (labelled G-SFT) with α optimized for homogeneous performance, and (iii) G-SFT with $\alpha \equiv 0.3$. We also compare to the inconsistent cell-centered DSA scheme of [2]. With our convergence criterion of 10^{-8} residual norm, source iteration by itself requires more than 10,000 iterations to converge each of Problems A-C.

In Problem A, c increases smoothly across the grid from $c_{\min}$ on the left to 0.9999 on the right. Σ_t is fixed and h is varied as specified in Table 1 to resolve a diffusion length. Since both the materials and grid vary extremely smoothly, SFTSA with $\alpha = 0.3$ converges. However, GMRES preconditioned with SFTSA performs better than all other methods except DSA.

Problem B is a two-material periodic medium for which performance of SFTSA preconditioned Krylov is the worst we have seen. The first cell is diffusive with optical thickness τ and the second cell is a pure absorber with optical thickness $1/\tau$. The heterogeneity is non-smooth, so SFTSA without Krylov diverges. Furthermore, it is not beneficial to precondition GMRES with SFTSA in any of the cases in Problem B.

In Problem C the optical thickness and scattering ratios of each cell are selected randomly within the constraints shown in Table 1. Despite the severity of the heterogeneity, the SFTSA preconditioner is effective with $\alpha = 0.3$.

CONCLUSIONS

For non-smooth heterogeneous problems, SFTSA without Krylov is unstable. However, Krylov preconditioned with SFTSA is stable and effective. The SFTSA preconditioner effectiveness does degrade, however, in the pathologically hard heterogeneous cases of Problems B and C. For realistic heterogeneities and reasonable grids, we expect SFTSA preconditioned Krylov to be significantly more effective than unpreconditioned Krylov when a large part of the problem is diffusive.

Although less efficient than DSA in this 1-D setting, SFTSA with Krylov has two major advantages over DSA. First, the stretch and filter equations are pure absorber transport problems, which

Table 1: Numerical Results: CPU Time Relative to a Single Source Iteration Sweep

	$c_{\min}$	SFT ($\alpha_{0.3}$)	G	G-SFT (α_{opt})	G-SFT ($\alpha_{0.3}$)	DSA
Prob. A	0	79	533	45	36	9
$c = (.9999 - c_{\min})j/J$	0.5	77	740	56	37	9
$\Sigma_t h = 0.1/\sqrt{1-c}$	0.9	73	1307	55	39	9
$1 \leq j \leq J = 10,000$	0.99	67	1297	47	37	9
	0.9999	54	113	62	30	6

	τ	SFT ($\alpha_{0.3}$)	G	G-SFT (α_{opt})	G-SFT ($\alpha_{0.3}$)	DSA
Prob. B	0.1	nc	7	19	16	8
Period-2	1	nc	14	67	54	10
$\Sigma_t h = [\tau,\ 1/\tau]$	10	nc	33	77	279	19
$c = [0.9999,\ 0]$	100	nc	20	40	43	15
Prob. C	0.1	49	81	32	30	11
Random	1	nc	96	94	53	15
$10^{-6} \leq \Sigma_t h \leq \tau$	10	nc	183	203	147	30
$0.95 \leq c \leq 0.9999$	100	nc	289	114	96	66

cost less time to solve (due to the lower quadrature order) than a single source iteration sweep. Second, the stretch and filter equations use the same spatial discretization (ES in this summary) as the underlying source iteration. Therefore, SFTSA can be used for problems where an effective DSA scheme has not been derived or where it is costly to solve the DSA diffusion equation. The second author, who originally developed the SFTSA method, has successfully implemented SFTSA in a 3-D unstructured grid code with a corner-balance spatial discretization [4] and has seen results similar to those we have presented for the ES scheme in 1-D. Hence, we believe that this method (SFTSA) is quite general in applicability, and relatively independent of the spatial discretization scheme.

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