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Modal Propagation in a Circular Tunnel with a Rough Wall: Surface-Impedance Formulation

K. F. Casey*

Abstract — **Propagation of waveguide modes in a circular tunnel with a rough wall is considered. An equivalent surface-impedance boundary condition is applied at a constant radius near the tunnel wall, obviating the need to consider the field in the surrounding lossy medium. The roughness of the interface is accommodated by making the equivalent surface impedance a random function of position on the wall. The scalar potential functions from which the field components are derived satisfy an infinite set of coupled homogeneous stochastic integral equations, approximate solutions to which are obtained by iteration. It is found that (1) the roughness of the wall permits the dominant mode for each azimuthal eigenfunction index to become slow when the frequency exceeds a certain critical value and (2) the power density carried by the incoherent part of the field tends to be concentrated near the rough wall. Representative numerical results are presented to illustrate the analysis.**

1 INTRODUCTORY REMARKS

Problems of wireless communication in tunnels are of current interest from both theoretical and practical points of view. A first step in addressing such problems is to develop a characterization of the propagation channel, including the effects of losses in the surrounding medium and of the roughness of the tunnel wall. In this paper we consider a representative problem of this class, specifically examining the propagation of leaky waveguide modes within a tunnel. Ray representations for the field within the tunnel are also under investigation, but are not discussed in this paper.

We assume that a quasi-circular tunnel is cut into a lossy surrounding medium. The interface between the tunnel interior and the outer medium is assumed to be rough. We obviate the need to consider the electromagnetic field in the exterior region through the use of an equivalent surface-impedance boundary condition that is defined at a constant radius $\rho = a$ within the tunnel. The roughness of the interface is accommodated by making the equivalent surface impedance a random function of the azimuthal and axial coordinates ϕ and z.

We begin by presenting an expression for the equivalent surface impedance $Z_s(\phi, z)$ in terms of the signal frequency, the relative permittivity of the outer medium, and the roughness of the wall itself. We assume that the tunnel wall is located at $\rho = a + \Delta(\phi, z)$ in which $\Delta(\phi, z)$ is a nonnegative homogeneous random function with prescribed (constant) mean $\overline{\Delta} = \mathcal{E}(\Delta)$ and autocovariance $C_\Delta(\phi - \phi', z - z') = \mathcal{E}\left\{\Delta(\phi, z)\Delta(\phi', z') - \overline{\Delta}^2\right\}$.

We assume the time dependence $\exp(j\omega t)$ for all field quantities. The equivalent surface impedance at the reference radius is expressed in terms of $\Delta(\phi, z)$ by

$$Z_s(\phi, z) = \frac{Z_0}{\sqrt{\epsilon_r}} + jk_0 Z_0 \left(1 - \frac{1}{\epsilon_r}\right) \Delta(\phi, z) \quad (1)$$

in which k_0 and Z_0 denote respectively the wavenumber and the intrinsic impedance of free space and ϵ_r is the (generally complex) relative permittivity of the outer medium. It is assumed that the phase change $k_0\Delta$ associated with the radial distance between the reference radius and the local wall radius is small compared to unity, and that the magnitude of the relative permittivity ϵ_r is large compared to unity.

The function $\Delta(\phi, z)$ is described by the Fourier-Stieltjes representation

$$\Delta(\phi, z) = \overline{\Delta} + \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{jm\phi} \int_{-\infty}^{\infty} e^{-jk_z z} d\nu_m(k_z) \quad (2)$$

in which

$$\mathcal{E}\left\{d\nu_m(k_z)\right\} = 0 \quad (3)$$

$$\mathcal{E}\left\{d\nu_m(k_z) d\nu_n^*(k_z')\right\} = 2\pi\delta_{mn}\delta(k_z - k_z')\mathcal{S}_m(k_z) dk_z dk_z' \quad (4)$$

The power spectral density functions $\mathcal{S}_m(k_z)$ are related to the autocovariance $C_\Delta(\phi - \phi', z - z')$ by

$$C_\Delta(\varphi, \zeta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\varphi} \cdot \int_{-\infty}^{\infty} \mathcal{S}_n(k_z) e^{-jk_z\zeta} dk_z \quad (5)$$

whence, by inverting the Fourier series and integral, we obtain

$$\mathcal{S}_m(k_z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} C_\Delta(\phi, z) e^{-jm\phi + jk_z z} d\phi dz \quad (6)$$

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The equivalent surface impedance, normalized by the intrinsic impedance of free space, is therefore described by the Fourier-Stieltjes representation

$$\zeta_s(\phi,z) = \frac{Z_s(\phi,z)}{Z_0} = \overline{\zeta}_s + jk_0\left(1-\frac{1}{\epsilon_r}\right)\cdot \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{jn\phi}\int_{-\infty}^{\infty} e^{-jk_z z} d\nu_m(k_z) \quad (7)$$

in which

$$\overline{\zeta}_s = \mathcal{E}\left\{\zeta_s(\phi,z)\right\} = \frac{1}{\sqrt{\epsilon_r}} + jk_0\left(1-\frac{1}{\epsilon_r}\right)\overline{\Delta} \quad (8)$$

2 FIELD COMPONENTS AND BOUNDARY CONDITIONS

The electromagnetic field within the tunnel is obtained from two scalar functions Φ and Ψ that satisfy the Helmholtz equation

$$\nabla^2 \begin{bmatrix}\Phi\\ \Psi\end{bmatrix} + k_0^2 \begin{bmatrix}\Phi\\ \Psi\end{bmatrix} = 0 \quad (9)$$

The electric and magnetic fields are given in terms of Φ and Ψ by

$$\vec{E} = \frac{Z_0}{jk_0}\nabla\times\nabla\times\Psi\vec{a}_z - \nabla\times\Phi\vec{a}_z \quad (10)$$

$$\vec{H} = \nabla\times\Psi\vec{a}_z + \frac{1}{jk_0 Z_0}\nabla\times\nabla\times\Phi\vec{a}_z \quad (11)$$

We represent the scalar functions Ψ (which generates the TM field components) and Φ (which generates the TE components) as

$$\begin{bmatrix} Z_0\Psi \\ \Phi\end{bmatrix}(\rho,\phi,z) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{jn\phi}\int_{-\infty}^{\infty}\cdot \begin{bmatrix}A_n(k_z)\\ B_n(k_z)\end{bmatrix} J_n(\lambda\rho)e^{-jk_z z}dk_z \qquad (\rho\le a) \quad (12)$$

in which $\lambda^2 = k_0^2 - k_z^2$ and the functions $A_n(k_z)$ and $B_n(k_z)$ are to be determined from the boundary conditions at the tunnel wall. These conditions are

$$E_\phi(a,\phi,z) = Z_0\zeta_s(\phi,z)H_z(a,\phi,z) \quad (13)$$

$$E_z(a,\phi,z) = -Z_0\zeta_s(\phi,z)H_\phi(a,\phi,z) \quad (14)$$

Substitution of the appropriate expressions obtained from equations (7), (10), and (11) yields an infinite set of coupled homogeneous stochastic integral equations for the functions $A_n(k_z)$ and $B_n(k_z)$. We define the column vector $\vec{A}_n(k_z)$ whose elements are $A_n(k_z)$ and $B_n(k_z)$ and the matrices $\mathbf{\Lambda}_n(k_z)$ and $\mathbf{\Gamma}_n(k_z)$ as follows:

$$\mathbf{\Lambda}_n(k_z) = \begin{bmatrix}\Lambda_{n11}(k_z) & \Lambda_{n12}(k_z)\\ \Lambda_{n21}(k_z) & \Lambda_{n22}(k_z)\end{bmatrix} \quad (15)$$

in which

$$\Lambda_{n11}(k_z) = \frac{nk_z}{jk_0 a}J_n(\lambda a) \quad (16)$$

$$\Lambda_{n12}(k_z) = \lambda J_n'(\lambda a) - \overline{\zeta}_s\frac{\lambda^2}{jk_0}J_n(\lambda a) \quad (17)$$

$$\Lambda_{n21}(k_z) = \frac{\lambda^2}{jk_0}J_n(\lambda a) - \overline{\zeta}_s\lambda J_n'(\lambda a) \quad (18)$$

$$\Lambda_{n22}(k_z) = \overline{\zeta}_s\frac{nk_z}{jk_0 a}J_n(\lambda a) \quad (19)$$

and

$$\mathbf{\Gamma}_n(k_z) = \begin{bmatrix} 0 & \lambda^2 J_n(\lambda a)\\ jk_0\lambda J_n'(\lambda a) & -\frac{nk_z}{a}J_n(\lambda a)\end{bmatrix} \quad (20)$$

Then the vectors $\vec{A}_n(k_z)$ satisfy the coupled homogeneous stochastic integral equations

$$\mathbf{\Lambda}_n(k_z)\cdot\vec{A}_n(k_z) = \left(1-\frac{1}{\epsilon_r}\right)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\cdot \mathbf{\Gamma}_{n-m}(k_z-k_z')\cdot\vec{A}_{n-m}(k_z-k_z')d\nu_m(k_z') \quad (21)$$

for $-\infty\le n\le\infty$ and $-\infty\le k_z\le\infty$.

3 APPROXIMATE SOLUTION OF THE COUPLED EQUATIONS

We develop an approximate solution to the coupled integral equations by iteration. We write

$$\vec{A}_n(k_z) = \vec{A}_n^{(0)}(k_z) + \vec{A}_n^{(1)}(k_z) + \cdots \quad (22)$$

The iteration scheme is

$$\mathbf{\Lambda}_n(k_z)\cdot\vec{A}_n^{(k)}(k_z) = \left(1-\frac{1}{\epsilon_r}\right)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\cdot \mathbf{\Gamma}_{n-m}(k_z-k_z')\cdot\vec{A}_{n-m}^{(k-1)}(k_z-k_z')d\nu_m(k_z') \quad (23)$$

for $k\ge 1$. The iteration is initialized using the condition

$$\mathbf{\Lambda}_n(k_z)\cdot\vec{A}_n^{(0)}(k_z) = 0 \quad (24)$$

In this work we consider the propagation of individual modes in the tunnel. The initialization of the iteration scheme is therefore selected so as to yield individual modes in the limit as the tunnel wall becomes smooth, that is, as the surface impedance becomes constant. The vanishing of the determinant of the matrix $\mathbf{\Lambda}_n(k_z)$ yields the dispersion relation

$$\left[\lambda a J_n'(\lambda a) - \overline{\zeta}_s\frac{(\lambda a)^2}{jk_0 a}J_n(\lambda a)\right]\cdot \left[\frac{(\lambda a)^2}{jk_0 a}J_n(\lambda a) - \overline{\zeta}_s\lambda a J_n'(\lambda a)\right] = \overline{\zeta}_s\left(\frac{nk_z a}{jk_0 a}\right)^2 J_n^2(\lambda a) \quad (25)$$

We denote the roots of this dispersion relation by $k_{zn\ell}$ for $\ell = 1, 2, \ldots$. We initialize the iteration at a specific root of the dispersion relation, say for $n = N$ and $k_z = k_{zN\ell}$, by writing

$$\vec{A}_n^{(0)}(k_z) = 2\pi \vec{A}_{N\ell} \delta_{nN} \delta(k_z - k_{zN\ell}) \tag{26}$$

in which $\vec{A}_{N\ell}$ is a constant vector whose components are related to each other by the condition

$$\mathbf{\Lambda}_N(k_{zN\ell}) \cdot \vec{A}_{N\ell} = 0 \tag{27}$$

The initialization condition (26) satisfies equation (24) for all $n \neq N$ and $k_z \neq k_{zN\ell}$ by virtue of the Kronecker and Dirac delta-functions in the definition of $\vec{A}_n^{(0)}(k_z)$; and it satisfies this equation for $n = N$ and $k_z = k_{zN\ell}$ because the dispersion relation is satisfied there.

The first iteration yields

$$\vec{A}_n^{(k)}(k_z) = \left(1 - \frac{1}{\epsilon_r}\right) \mathbf{\Lambda}_n^{-1}(k_z) \cdot \mathbf{\Gamma}_N(k_{zN\ell}) \cdot \vec{A}_{N\ell} \int_{-\infty}^{\infty} \delta(k_z - k_z' - k_{zN\ell}) d\nu_{n-N}(k_z') \tag{28}$$

We thus obtain the following first-order expression for the scalar functions Ψ and Φ:

$$\begin{bmatrix} Z_0 \Psi(\rho, \phi, z) \\ \Phi(\rho, \phi, z) \end{bmatrix} = e^{jN\phi - jk_{zN\ell}z} J_N(\lambda_{N\ell}\rho) \vec{A}_{N\ell} + \left(1 - \frac{1}{\epsilon_r}\right) \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{j(m+N)\phi} \int_{-\infty}^{\infty} e^{-j(k_z + k_{zN\ell})z} J_{m+N}(\lambda'\rho) \mathbf{\Lambda}_{m+N}^{-1}(k_z' + k_{zN\ell}) \cdot \mathbf{\Gamma}_N(k_{zN\ell}) \cdot \vec{A}_{N\ell}\, d\nu_m(k_z') \tag{29}$$

in which $\lambda' = \sqrt{k_0^2 - (k_z' + k_{zN\ell})^2}$. This completes the first-order approximate solution for the functions from which the electromagnetic field components are derived. These functions each comprise a deterministic or coherent part and a zero-mean random or incoherent part; and so therefore will also the field components derived from them.

4 THE DISPERSION RELATION

It is evident from the dispersion relation that the modes are in general hybrid unless $n = 0$ or $\overline{\zeta}_s = 0$. All the modes are leaky unless the outer medium becomes perfectly conducting. Investigation of the dispersion relation reveals that for each value of n the dominant mode becomes slow (that is, the axial propagation constant k_z exceeds the free-space wavenumber k_0) when the frequency exceeds a certain limiting value. The limiting frequency is given for $n = 0$ by

$$(k_0 a)^2 = \frac{2a}{\overline{\Delta}} \tag{30}$$

and for $n > 0$ by

$$(k_0 a)^2 = \frac{a(n+1)(na + \overline{\Delta})}{n\overline{\Delta}[a + (n+1)\overline{\Delta}]} \approx \frac{(n+1)a}{\overline{\Delta}} \tag{31}$$

All the higher-order modes for that value of n remain fast.

The existence of the slow-wave modes is a consequence of the inductive component of the equivalent surface impedance that arises from the roughness of the the wall and that is evident in the second term on the RHS of equations (1) and (8). The inductive reactance can bind a wave to the rough surface of the tunnel; the bound wave decays approximately exponentially away from the tunnel wall.

Representative numerical results are shown in Figure 1. The relative permittivity of the surrounding medium is given by [1], [2]

$$\epsilon_r = \left(\sqrt{\epsilon_{r\infty}} + \sqrt{\sigma_0 Z_0/(jk_0)}\right)^2 \tag{32}$$

in which $\epsilon_{r\infty}$ is the high-frequency relative permittivity and σ_0 is the low-frequency conductivity. We

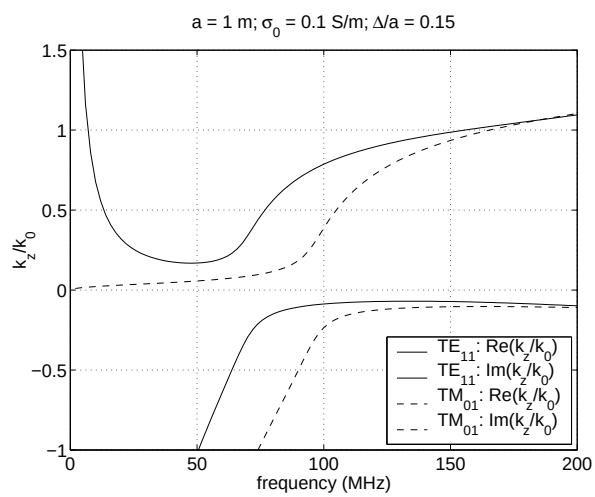

Figure 1: Normalized propagation constant for lowest-order modes vs. frequency: $\overline{\Delta} = 0.15a$.

show the real and imaginary parts of the normalized complex propagation constant k_z/k_0 as functions of frequency for the two lowest-order modes in a tunnel of radius one meter. (These modes reduce to the TE_{11} and TM_{01} modes in the limit as the tunnel wall becomes smooth and perfectly conducting.) The average roughness $\overline{\Delta}$ is equal to $0.15a$; and $\epsilon_{r\infty} = 7.0$ and $\sigma_0 = 0.1$ S/m. The real parts of the propagation constant are positive and the imaginary parts are negative. The cutoff frequencies are approximately 75 MHz for the "TE_{11}" mode and approximately 100 MHz for the " TM_{01}" mode.

The modes become slow at frequencies somewhat above 150 MHz.

5 AXIAL POWER DENSITY

The axial power density $S_z(\rho, \phi, z)$ is given in general by

$$S_z(\rho, \phi, z) = \frac{1}{2}\Re\left[E_\rho H_\phi^* - E_\phi H_\rho^*\right] \qquad (33)$$

The expected value of the axial power density for a given mode is the sum of the coherent and incoherent power densities. The coherent axial power density is given by

$$S_{zc} = \frac{|A_{N\ell}|^2 + |B_{N\ell}|^2}{2Z_0\rho^2}\frac{k_{zN\ell r}}{k_0}e^{2k_{zN\ell i}z}. \left[N^2|J_N(\lambda\rho)|^2 + |\lambda\rho J_N'(\lambda\rho)|^2\right] \qquad (34)$$

in which $\lambda^2 = k_0^2 - k_{zN\ell}^2$; subscripts indicate the real and imaginary parts of the axial propagation constant $k_{zN\ell}$. The incoherent axial power density is given by

$$S_{zi} = \frac{e^{2k_{zN\ell i}z}}{2\pi Z_0\rho^2}\left|1 - \frac{1}{\epsilon_r}\right|^2 \cdot \sum_{m=-\infty}^{\infty}\int_{\infty}^{\infty} Q_{m,N}(\rho; k_z'; k_{zN\ell})\mathcal{S}_m(k_z')dk_z' \qquad (35)$$

in which $Q_{m,N}(\rho; k_z'; k_{zN\ell})$ is a rather complicated function of m, N, ρ, k_z', and $k_{zN\ell}$. We shall not take the space to write it out. We remark that these power densities are independent of the azimuthal coordinate ϕ, since we have considered the angular eigenfunctions $\exp(jn\phi)$, rather than superpositions of these such as $\sin n\phi$ and $\cos n\phi$.

The radial dependence of $Q_{m,N}(\rho; k_z'; k_{zN\ell})$ arises from the presence of terms involving the Bessel function $J_{m+N}(\lambda'\rho)$ and its derivative. We consider situations in which the wavenumbers that characterize the wall roughness are large in comparison to those that describe the modal field: that is, the power spectral density functions $\mathcal{S}_m(k_z')$ are significantly different from zero only for $|m| >> |N|$ and $|k_z'| >> |k_{zN\ell}|$. In these situations, the radial dependence of the incoherent axial power density is approximately exponential; thus the incoherent axial power density tends to be confined to the vicinity of the rough wall. We show an example in Figure 2, in which we plot the incoherent axial power density, normalized to its value at the tunnel wall, vs. normalized radial position ρ/a. The parameters are those used for Figure 1 (TE_{11} mode) and the frequency is 120 MHz. In this example case, the incoherent axial power density is effectively zero for $0 \le \rho \le 0.6a$.

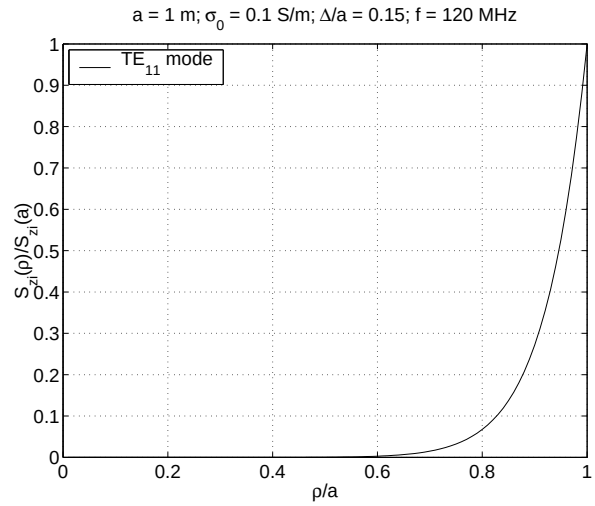

Figure 2: Normalized incoherent axial power density vs. ρ/a: $\overline{\Delta} = 0.15a$, $f = 120$ MHz, TE_{11} mode.

6 CONCLUDING REMARKS

We have formulated the problem of electromagnetic wave propagation in a circular tunnel with a rough wall, using an equivalent surface-impedance boundary condition. Using an iterative technique, we have obtained an approximate solution to the coupled homogeneous stochastic integral equations that describe the propagation. Under an appropriate initialization, our approximate solution represents single leaky waveguide modes. We have shown that slow-wave modes can exist when the wall is rough and that the incoherent part of the axial power density is confined to the vicinity of the rough wall.

We can now develop the covariance functions of the electromagnetic field in the tunnel. Arguments based on the central limit theorem can be used to show that the field is Gaussian; the covariance functions will therefore complete the description of the probabilistic character of the field.

References

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[2] F. M. Tesche, "On the modeling and representation of a lossy earth for transient electromagnetic field calculations", *Theoretical Notes*, Note 367, July 2002.

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