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Stability of stationary solutions of the multifrequency radiation diffusion equations

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Abstract

A nondimensional model of the multifrequency radiation diffusion equation is derived. A single material, ideal gas, equation of state is assumed. Opacities are proportional to the inverse of the cube of the frequency. Inclusion of stimulated emission implies a Wien spectrum for the radiation source function. It is shown that the solutions are uniformly bounded in time and that stationary solutions are stable. The spatially independent solutions are asymptotically stable, while the spatially dependent solutions of the linearized equations approach zero.

1 Introduction

This paper derives asymptotic stability properties for a system of equations modeling the frequency dependent radiation diffusion equation coupled to the matter energy balance equation. In order to obtain a tractable system, we simplify the equations and first derive a system in nondimensional form. Our derived system preserves the salient features of the original, dimensional set of equations; in particular, frequency dependent opacities and a nonlinear relationship between the matter temperature and the radiation emission term. We assume matter to consist of a single material characterized by an ideal gas equation-of-state, i.e., a specific energy proportional to temperature. In the analysis, we impose zero flux boundary conditions on the radiation field. However, since we are interested in studying how the two fields (radiation and temperature) equilibrate, and since the radiation diffusion equation is itself derived assuming near isotropy, i.e., short mean free paths, our boundary condition is not overly restrictive.

We begin with the multifrequency radiation diffusion equations (CGS units),

$$\partial_t u_\nu = \nabla D_\nu \nabla u_\nu + c\,\kappa_\nu\,(\,B_\nu - u_\nu\,)\,, \tag{1}$$

$$\rho \partial_t e = -\,c \int_0^\infty d\nu\,\kappa_\nu\,(\,B_\nu - u_\nu\,)\,. \tag{2}$$

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Equations (1)–(2) simulate diffusive transport and emission-absorption. The equations do not contain scattering nor effects due to fluid motion (advection and spectral shifts.) The variable u_ν (erg sec cm^{-3}) represents the spectral energy density of the radiation field, ν (sec^{-1}) is the frequency coordinate, ρ is the mass density (assumed fixed), and e is the matter specific energy density. The latter's temporal derivative is expressed in terms of the specific heat $c_{\rm v}$ and the matter temperature T,

$$\partial_t e = \frac{\partial e}{\partial T}\,\partial_t T \doteq c_{\rm v}\partial_t T \ . \tag{3}$$

On the right side of (1), the diffusion coefficient $D_\nu = c/3\kappa_\nu$, where c is the speed of light and κ_ν (cm^{-1}) is the inverse mean free path.[1] (In most applications, D_ν is modified by including the total scattering cross-section, (Mihalas and Mihalas [9]) and a flux limiter (Lund and Wilson [7].) Section 6 discusses how such generalizations impact our model. The coefficient κ_ν is a complicated function of frequency. In this paper, we consider one valid for free-free transitions [12],

$$\kappa_\nu = (2\pi/3m_e^3k)^{1/2}\,(4Z_e^2e^6N_+N_e/3hc)\,T^{-1/2}\,\nu^{-3}\ , \tag{4}$$

where m_e is the electron mass, k is the Boltzmann constant, Z_e is the ionic charge, e is the fundamental charge, N_+ and N_e are the number densities (# cm^{-3}) of the ions and free electrons, and h is the Planck constant. We focus attention on a fully ionized hydrogen plasma of density ρ. Thus, $Z_e = 1$, and $N_+ = N_e = \rho\mathcal{A}$ where $\mathcal{A}$ is the Avogadro constant. This implies that

$$\kappa_\nu = \kappa_0\ \nu^{-3}\ \ ,\quad \kappa_0 = 1.34\cdot 10^{56}\,\rho^2/\sqrt{T}\ . \tag{5}$$

In order to obtain a tractable set of equations, we ignore the weak temperature dependence of κ_0 and, since ρ is assumed to be fixed, absorb $\rho^2/\sqrt{T}$ into κ_0.[2] Thus, taking characteristic values for ρ and T, we let $\kappa_0\rho^2/\sqrt{T} \to \kappa_0$. Since κ_ν has units of cm^{-1}, κ_0 now has units of cm^{-1} sec^{-3}.

Lastly, we define the radiation emission term B_ν. To simplify the algebra, instead of the usual Planck function we assume a Wien distribution and write,

$$B_\nu = (8\pi h/c^3)\,\nu^3\ \exp(-h\nu/kT)\ . \tag{6}$$

The difference between (6) and the Planck function is that the latter replaces the exponential term with $[\exp(h\nu/kT)-1]^{-1}$. For our purposes, this is of little consequence. Both distributions, Planck and Wien, have the same exponential decay at high frequencies. At low frequencies, (6) and the Planck function are proportional to ν^3 and ν^2, respectively. Equation (6) and the Planck function peak at $h\nu/kT = 3$ and 2.82, respectively. Also, the frequency integrated emission term

$$\int_0^\infty d\nu\,B_\nu = 7.00\cdot 10^{-15}\,T^4\ ,$$

[1] Some authors express the mean free path as $(\rho\kappa_\nu)^{-1}$ where κ_ν (cm^2/g) is the opacity.

[2] Section 6 discusses the effect of keeping the $1/\sqrt{T}$ dependence of κ_ν.

which is approximately 8% less than the Planck result, $7.56 \cdot 10^{-15}\, T^4$. Lastly, the relative error of the two distributions, (Planck − Wien)/Planck = $\exp(-h\nu/kT)$. Thus, the error is greatest at $\nu = 0$, but at the peak of the Planck function, the error is less than 6%.

We now derive the nondimensional system of equations and begin by choosing characteristic values for the density ρ, specific heat $c_{\rm v}$, and temperature T_0. As mentioned above, incorporating these values into (5) yields $\kappa_\nu = \kappa_0 \nu^{-3}$. Denoting the normalization values with zero subscripts and the normalized variables with primes, we define, $\nu' = \nu/\nu_0$, $T' = T/T_0$, $u' = u/u_0$, $t' = t/t_0$, and if x is the spatial coordinate, $x' = x/x_0$. This implies that $\kappa_\nu = \kappa_0(\nu_0\nu')^{-3}$, $\partial_t = (1/t_0)\,\partial_{t'}$, $\nabla = (1/x_0)\nabla'$, etc. The constants ν_0, t_0, etc., may be expressed in terms of the original three, ρ, $c_{\rm v}$, and T_0. The choices,

$$\begin{aligned}
\nu_0 &= kT_0/h\,, \\
t_0 &= \nu_0^3/(c\,\kappa_0)\,, \\
x_0 &= \nu_0^3/(\sqrt{3}\,\kappa_0)\,, \\
u_0 &= 8\pi\, h\, \nu_0^3/c^3\,, \\
R &= \rho\, c_{\rm v}\, c^3/(8\pi\, k\, \nu_0^3)\,,
\end{aligned} \tag{7}$$

yield the desired system,

$$\partial_t u = \nabla \cdot \nu^3\, \nabla u + (\nu^3 e^{-\nu/T} - u)\,/\,\nu^3\,, \tag{8}$$

$$R\partial_t T = -T + \int_0^\infty (u/\nu^3)\, d\nu\,, \tag{9}$$

where $u = u(x,t,\nu)$ and $T = T(x,t)$. In (8)–(9) and henceforth, we drop the primes from the nondimensional variables, t, ν, u, etc. Note that the first term on the right side of (9) stems from integrating over frequency, i.e.,

$$\int_0^\infty d\nu\, (\nu^3 e^{-\nu/T})/\nu^3 = \int d\nu\, e^{-\nu/T} = T\,.$$

It is instructive to choose characteristic values for T_0, ρ, and $c_{\rm v}$, and evaluate the normalization constants in (7). For example, the choices $T_0 = 10^5$ °K, $\rho = 10^{-6}$ g cm^{-3}, and $c_{\rm v} = 10^8$ erg/(g °K), imply $\nu_0 = 2.08 \cdot 10^{15}$ Hz ($\approx$ 10 eV), $t_0 = 7.12 \cdot 10^{-7}$ sec, $x_0 = 1.24 \cdot 10^4$ cm, $u_0 = 5.59 \cdot 10^{-11}$ erg sec/cm^3, and $R = 85.8$ Note that the above value for ν_0 corresponds to a wavelength $\lambda_0 = 1.44 \cdot 10^{-5}$ cm, which lies just outside the so-called near UV range: 2–4 $\cdot\, 10^{-5}$ cm.

The nondimensional equations (8)–(9) preserve important properties of the original system (1)–(2). Conservation of total energy follows by ignoring boundary fluxes (e.g., by imposing homogeneous Neumann boundary conditions on u) and integrating the total energy density $RT + \int d\nu\, u$ over the spatial domain. The high frequency photons, i.e., u with large ν in (8)–(9) are characterized by fast transport and slow absorption. In (8), $1/\nu^3$ plays the role of a coupling coefficient; the low frequencies coupling

the fastest. Matter preferentially emits radiation into frequencies where $\nu^3\, e^{-\nu/T}$ is maximal.

Before launching into our theme, we summarize previously published, related work. Andreev et al [1] consider the system coupling the matter energy equation to either the equation of radiative transfer (i.e., for the radiation intensity I) or to the radiation diffusion equation. They examine the grey, i.e., frequency-integrated, as well as the spectral cases. Andreev et al assume Dirichlet boundary conditions; that is, the incoming intensity is specified, or in the diffusion limit, the radiation energy density. Isotropic scattering is included in the transfer equation. Assuming existence of solutions, the paper proves a maximum principle, viz., that the solution is bounded by the initial data and the boundary values. The paper does not discuss whether the equilibrium solution is an attractor.

Mercier [8] considers the system coupling the spectral radiative transfer and matter energy equations. The system includes Thomson scattering, i.e., no energy exchange for the scattering process. By applying the theory of Accretive Operators, Mercier shows that: (1) A solution exists. (2) The solution is bounded by the initial and boundary data. (3) As time progresses, the distance between the solution and the equilibrium point decreases, i.e., the latter is an attractor. Mercier makes the following assumptions regarding the behavior of the coefficients of the equations: The matter internal energy has a positive minimum and the specific heat, a finite upper bound. For the opacity κ, Lipschitz continuity with respect to T, a finite upper bound for all frequency, and that κ decreases with T. Lastly, that radiation emission $\kappa\, B$ increases with T. These assumptions are largely satisfied by our model, with the exceptions that our κ diverges with decreasing frequency (as ν^{-3}) and that κ is independent of T.

The existence of a maximum principle (MP) has implications for the design of numerical methods for these equations. Ideally, the schemes should also satisfy a MP. In [6], Larsen and Mercier apply the theoretical work of [1] and [8] to analyze the Monte Carlo method of Fleck and Cummings [5]. Larsen and Mercier prove that for sufficiently small time steps, the algorithm does satisfy the MP, but that it fails to do so for large time steps. The analysis is confirmed by their numerical simulations and by Fleck and Cummings' own results [5], Fig. 4, p. 332.

There are two competing Monte Carlo schemes for the equations of radiative transfer. One is the previously mentioned one of Fleck and Cummings [5]; another is the method of Carter and Forest [2]. Both schemes rewrite the matter energy equation in terms of the integrated radiation energy density function $u_r(T) = aT^4$, where a is the radiation constant. The schemes require a value of u_r throughout the course of the time step, but they differ in how that term is approximated. The choice of approximation impacts the stability of the scheme, i.e., whether or not the numerical solution satisfies the maximum principle. In [3], Densmore and Larsen apply the Larsen and Mercier analysis [6] to the Carter-Forest scheme. Densmore and Larsen's analysis shows that the Fleck-Cummings scheme appears to allow larger time steps than Carter-Forest before the numerical solution violates the maximum principle. The analysis is confirmed by numerical results on a problem simulating radiative flow into initially cold matter.

In this paper, we extend the work just cited by analyzing the rate at which solutions equilibrate. Since our interests lie with the multifrequency radiation diffusion equations, we restrict attention to the normalized system derived above.

We now summarize the rest of the paper. Section 2 confirms that solutions are positive and uniformly bounded by the initial conditions. Section 3 shows that the stationary solution is unique and if the initial conditions of a given problem lie within an envelope about the stationary solution, so does the solution for all time. In section 4, we consider a linearized form of (8)–(9) and prove that a small perturbation about the stationary solution decays to zero exponentially fast, if the perturbation is spatially varying. Section 5 considers the nonlinear system (8)–(9) and shows that non spatially varying perturbations also decay to zero. However, we are unable to give an estimate for the rate of decay and surmise that the decay may be slow. A summary, motivation and for this work, and description of ongoing, related research appear in section 6. Section 6 also discusses the impact that more realistic assumptions, e.g., non-ideal gases, have on our model.

2 A priori bounds

Let U be an open, bounded and connected subset of $\mathbb{R}^3$ with smooth boundary. We consider the equations

$$\partial_t u(x,t,\nu) + \frac{1}{\nu^3} u(x,t,\nu) = \nu^3 \Delta u(x,t,\nu) + e^{-\nu/T(x,t)}, \tag{10}$$

$$R\,\partial_t T(x,t) + T(x,t) = \int_{\nu=0}^{\infty} \frac{u(x,t,\nu)}{\nu^3} d\nu, \tag{11}$$

for $x \in U$, $t > 0$ and $0 < \nu < \infty$ with boundary conditions

$$\frac{\partial u}{\partial n}(x,t,\nu) = 0, \qquad x \in \partial U \tag{12}$$

and initial conditions

$$u(x,0,\nu) = u_0(x,\nu), \qquad T(x,0) = T_0(x), \qquad x \in U\ .$$

Here $T(x,t)$ is the matter temperature at position x and time t, $u(x,t,\nu)$ denotes the radiation energy density at frequency ν, and $\Delta u = \sum_{i=1}^{3} \partial^2_{x_i} u$. The constant R is positive.

Lemma 1 *Let $u(x,t,\nu)$, $T(x,t)$ be smooth solutions of (10)-(12). If*

$$0 \le u(x,0,\nu) \le \nu^3 e^{-\nu/T_2} \qquad \textit{and} \qquad 0 < T(x,0) < T_2,$$

then, for $t > 0$

$$0 < u(x,t,\nu) < \nu^3 e^{-\nu/T_2} \qquad \textit{and} \qquad 0 < T(x,t) < T_2.$$

Remark Lemma 1 shows that the solutions of (10)–(12) are uniformly bounded in time. This is to be expected as the temperature and the radiation energy are non-negative quantities and there is no energy flowing through the boundary. The result is also valid when U is a box with periodic boundary conditions.

Proof The idea is to rewrite (10)–(11) as integral equations, and use proof by contradiction. It follows from (10) that for each $\nu > 0$,

$$\partial_t (e^{t/\nu^3} u) = \nu^3 \Delta (e^{t/\nu^3} u) + e^{t/\nu^3} e^{-\nu/T} \ .$$

Using Duhamel's principle [4], p.49, we obtain

$$\begin{aligned} u(x,t,\nu) &= e^{-t/\nu^3} \int_U G_\nu(x,y,t)\, u(y,0,\nu) dy \\ &\quad + \int_{s=0}^t e^{-(t-s)/\nu^3} \int_U G_\nu(x,y,t-s) e^{-\nu/T(y,s)} dy\, ds \ . \end{aligned} \tag{13}$$

Here $G_\nu(x,y,t)$ is the Green's function for $\partial_t w = \nu^3 \Delta w$ with Neumann boundary conditions. From (11) we see that

$$\partial_t (e^{t/R} T) = \frac{e^{t/R}}{R} \int_{\nu=0}^\infty \frac{u}{\nu^3} d\nu.$$

Integrating with respect to t and using (13) yields

$$\begin{aligned} T(x,t) &= e^{-t/R}\, T(x,0) \\ &\quad + \int_{\tau=0}^t \frac{e^{-(t-\tau)/R}}{R} \int_{\nu=0}^\infty \frac{e^{-\tau/\nu^3}}{\nu^3} \int_U G_\nu(x,y,\tau) u(y,0,\nu) dy\, d\nu\, d\tau \\ &\quad + \int_{\tau=0}^t \frac{e^{-(t-\tau)/R}}{R} \int_{\nu=0}^\infty \int_{s=0}^\tau \frac{e^{-(\tau-s)/\nu^3}}{\nu^3} \int_U G_\nu(x,y,\tau-s) e^{-\nu/T(y,s)} dy ds d\nu d\tau. \end{aligned} \tag{14}$$

To bound T we assume that either $\min_{\bar{U}} T(x,t) = 0$ or $\max_{\bar{U}} T(x,t) = T_2$ for some $t > 0$. Let t_2 be the smallest such t. Then $0 < T(x,t) < T_2$ for $0 \le t < t_2$. Since $T_0 > 0$, $G_\nu > 0$, $u_0 \ge 0$, $R > 0$, and $e^{-\nu/T} > 0$ for $t < t_2$ we conclude from (14) that $T(x,t_2) > 0$. Thus, the upper bound must hold at $t = t_2$. Let x_2 define the point where $T(x_2,t_2) = T_2$. Recall that $\int G_\nu dy \equiv 1$. Since $u(y,0,\nu) \le \nu^3 e^{-\nu/T_2}$

and $e^{-\nu/T(y,s)} < e^{-\nu/T_2}$ for $s < t_2$ it follows from (14), after integrating over s, that

$$\begin{aligned} T_2 = T(x_2,t_2) \le e^{-t_2/R} T(x_2,0) \\ &+ \int_{\tau=0}^{t_2} \frac{e^{-(t_2-\tau)/R}}{R} \int_{\nu=0}^{\infty} e^{-\tau/\nu^3} e^{-\nu/T_2} d\nu d\tau \\ &+ \int_{\tau=0}^{t_2} \frac{e^{-(t_2-\tau)/R}}{R} \int_{\nu=0}^{\infty} (1-e^{-\tau/\nu^3}) e^{-\nu/T_2} d\nu d\tau \,. \end{aligned}$$

Cancelling the terms involving $e^{-\tau/\nu^3}$ and evaluating the remaining integrals yields the contradiction,

$$T_2 \le e^{-t_2/R}\,[T(x_2,0) - T_2] + T_2 < T_2 \,.$$

Thus, the bounds for $T(x,t)$ hold for all time.

To bound $u(x,t,\nu)$, we let $t > 0$ be given and choose $\epsilon > 0$ such that

$$\epsilon < \min_{y\in\bar{U},\ 0\le s\le t} T(y,s) \le \max_{y\in\bar{U},\ 0\le s\le t} T(y,s) < T_2 - \epsilon \,.$$

Since $u(x,0,\nu) \ge 0$, $G_\nu > 0$, and $\int G_\nu dy \equiv 1$, it follows from (13) that $u(x,t,\nu) \ge (1-e^{-t/\nu^3})e^{-\nu/\epsilon} > 0$, which establishes the lower bound. For the upper bound, we use the inequalities $u(x,0,\nu) \le \nu^3 e^{-\nu/T_2}$ and $e^{-\nu/T(y,s)} < e^{-\nu/(T_2-\epsilon)}$. Doing the integrals in (13) gives

$$\begin{aligned} u(x,t,\nu) &\le e^{-t/\nu^3}\nu^3 e^{-\nu/T_2} + (1-e^{-t/\nu^3})\nu^3 e^{-\nu/(T_2-\epsilon)} \\ &\le \nu^3 e^{-\nu/T_2} - (1-e^{-t/\nu^3})\nu^3 e^{-\nu/T_2}\big(\,1 - e^{-\nu\epsilon/[T_2(T_2-\epsilon)]}\,\big) \\ &< \nu^3 e^{-\nu/T_2} \,. \end{aligned}$$

This completes the proof.

3 Stationary solutions

In this section we consider stationary solutions of (10)–(12). We shall show that such solutions cannot depend on x, and that they are stable. The problem of asymptotic stability is discussed in sections 4 and 5.

Lemma 2 *Let $u(x,\nu)$, $T(x)$ be a stationary solution of (10)–(12) and satisfy the bounds in Lemma 1. Then u, T are independent of x and there is a constant M such that*

$$u(\nu) = \nu^3 e^{-\nu/M}, \qquad T = M \,.$$

Proof It is clear that $u = \nu^3 e^{-\nu/M}$, $T = M$ is a stationary solution of (10)–(12). We will show that there are no others. It follows from (10)–(12) that

$$-\nu^6 \Delta u(x,\nu) + u(x,\nu) = \nu^3 e^{-\nu/T(x)} ,$$
$$T(x) = \int_{\nu=0}^{\infty} \frac{u(x,\nu)}{\nu^3} d\nu , \tag{15}$$

where u satisfies the boundary condition (12). Since $-\nu^6 \Delta w + w = f$ with $\partial w/\partial n = 0$ has a unique solution and $w = 1$ if $f = 1$, we see that

$$u(x,\nu) = \int_U g_\nu(x,y) \nu^3 e^{-\nu/T(y)} dy , \tag{16}$$
$$1 = \int_U g_\nu(x,y) dy . \tag{17}$$

One can show that the Green's function satisfies $g_\nu(x,y) = g_\nu(y,x) > 0$ when $x \neq y$ and is singular when $x = y$. Let $M = \max_{\bar{U}} T(y)$. Then $M = T(x)$ for some x in $\bar{U}$ and by using (17), (15), and (16) we find,

$$\begin{aligned} 0 &= M - T(x) \\ &= \int_{\nu=0}^{\infty} e^{-\nu/M} \int_U g_\nu(x,y) dy d\nu - \int_{\nu=0}^{\infty} \frac{1}{\nu^3} \int_U g_\nu(x,y) \nu^3 e^{-\nu/T(y)} dy d\nu \\ &= \int_{\nu=0}^{\infty} \int_U g_\nu(x,y) \left[e^{-\nu/M} - e^{-\nu/T(y)} \right] dy d\nu . \end{aligned}$$

Since $g_\nu > 0$ and $[e^{-\nu/M} - e^{-\nu/T(y)}] \geq 0$, we conclude that $T(y) = M$ for all y in U. Equations (16)–(17) then imply that $u(x,\nu) = \nu^3 e^{-\nu/M}$ for all x in U. This completes the proof.

How do we determine the stationary solution corresponding to particular initial data? Let

$$K(t) = \int_U \left[RT(x,t) + \int_{\nu=0}^{\infty} u(x,t,\nu) d\nu \right] dx , \tag{18}$$

define the total (matter and radiation) energy for the system. Differentiating with respect to t and using (10)–(12) we see that $K(t)$ is independent of time. If $T(x,t) \to M$ and $u(x,t,\nu) \to \nu^3 e^{-\nu/M}$ as $t \to \infty$ we can find M by solving

$$RM + 6M^4 = K(0) \,/\, \mathrm{vol}(U) . \tag{19}$$

If $K(0) > 0$ this equation has a unique solution. We will now show that the stationary solution is stable.

Theorem 1 *Let $u(x,t,\nu)$, $T(x,t)$ be a smooth solution of (10)–(12). If*

$$\nu^3 e^{-\nu/T_1} \leq u(x,0,\nu) \leq \nu^3 e^{-\nu/T_2} \qquad and \qquad T_1 < T(x,0) < T_2,$$

then for $t > 0$

$$\nu^3 e^{-\nu/T_1} < u(x,t,\nu) < \nu^3 e^{-\nu/T_2} \qquad and \qquad T_1 < T(x,t) < T_2.$$

Remark Equations (18)–(19) show that if u and T satisfy the assumptions in Theorem 1 then $RT_1 + 6T_1^4 < RM + 6M^4 < RT_2 + 6T_2^4$. Therefore, the temperature M for the stationary solution lies in the interval $T_1 < M < T_2$.

Proof We have already established the upper bounds in Lemma 1. Our proof for the lower bounds is similar. Assume $\min_{\bar{U}} T(x,t) = T_1$ for some $t > 0$ and let t_1 be the first such t. Then, $T_1 = T(x_1, t_1)$ for some $x_1 \in \bar{U}$ and $T(y,s) > T_1$ for $y \in U$ and $s < t$. Since $\int G_\nu dy \equiv 1$, $u(y,0,\nu) \geq \nu^3 e^{-\nu/T_1}$ and $e^{-\nu/T(y,s)} > e^{-\nu/T_1}$ for $s < t_1$, it follows from (14), after integrating over s, that

$$\begin{aligned} T_1 = T(x_1,t_1) \geq\ & e^{-t_1/R} T(x_1,0) \\ & + \int_{\tau=0}^{t_1} \frac{e^{-(t_1-\tau)/R}}{R} \int_{\nu=0}^{\infty} e^{-\tau/\nu^3} e^{-\nu/T_1} d\nu d\tau \\ & + \int_{\tau=0}^{t_1} \frac{e^{-(t_1-\tau)/R}}{R} \int_{\nu=0}^{\infty} (1 - e^{-\tau/\nu^3}) e^{-\nu/T_1} d\nu d\tau \,. \end{aligned}$$

Cancelling the $e^{-\tau/\nu^3}$ terms and evaluating the remaining integrals we get

$$T_1 \geq e^{-t_1/R} \left[T(x_1,0) - T_1 \right] + T_1 > T_1 \,.$$

The contradiction shows that the lower bound for $T(x,t)$ holds for all time.

To bound $u(x,t,\nu)$ from below we let $t > 0$ be given and choose $\epsilon > 0$ such that

$$T_1 + \epsilon < \min_{y \in \bar{U},\ 0 \leq s \leq t} T(y,s) \,.$$

Since $u(x,0,\nu) \geq \nu^3 e^{-\nu/T_1}$ and $e^{-\nu/T(y,s)} > e^{-\nu/(T_1+\epsilon)}$ it follows from (13) that

$$\begin{aligned} u(x,t,\nu) &\geq e^{-t/\nu^3} \nu^3 e^{-\nu/T_1} + (1 - e^{-t/\nu^3}) \nu^3 e^{-\nu/(T_1+\epsilon)} \\ &\geq \nu^3 e^{-\nu/T_1} + (1 - e^{-t/\nu^3}) \nu^3 e^{-\nu/(T_1+\epsilon)} (1 - e^{-\nu\epsilon/[T_1(T_1+\epsilon)]}) \\ &> \nu^3 e^{-\nu/T_1} \,. \end{aligned}$$

This completes the proof.

4 Linearized equations

In section 3 we have shown that (10)–(12) has a stable stationary solution. The important question is whether the stationary solution is asymptotically stable, i.e., will nearby solutions tend to the stationary solution as time increases. We approach this from two directions. In this section we linearize (10)–(12) around the stationary solution and prove that the linearized equation is asymptotically stable when the initial data are orthogonal to constant functions. In the next section we show that (10)–(12) is asymptotically stable if the initial data are expressed in terms of constant functions.

To get the linearized equations we assume that $u = \nu^3 e^{-\nu/M} + u'$ and $T = M + T'$ where u' and T' are small. Inserting in (10)–(12), expanding $e^{-\nu/T}$ in a Taylor series, and discarding the higher order terms lead to the linearized equations,

$$\partial_t u'(x,t,\nu) + \frac{1}{\nu^3} u'(x,t,\nu) = \nu^3 \Delta u'(x,t,\nu) + \frac{\nu e^{-\nu/M}}{M^2} T'(x,t), \tag{20}$$

$$R\,\partial_t T'(x,t) + T'(x,t) = \int_{\nu=0}^{\infty} \frac{u'(x,t,\nu)}{\nu^3} d\nu, \tag{21}$$

where u' satisfies the original boundary condition (12). Since M satisfies (19), we conclude from (18) that for every $t \geq 0$,

$$\int_U \left[R\,T'(x,t) + \int_{\nu=0}^{\infty} u'(x,t,\nu) d\nu \right] dx = 0\,,$$

We solve (20)–(21) by separation of variables. Let

$$u'(x,t,\nu) = \sum_{k=0}^{\infty} a_k(t,\nu) e_k(x), \qquad T'(x,t) = \sum_{k=0}^{\infty} \alpha_k(t) e_k(x)\,, \tag{22}$$

where $\Delta e_k + \lambda_k e_k = 0$ in U, $(\partial/\partial n) e_k = 0$ on ∂U and $\int_U e_i e_j dx = \delta_{ij}$. It can be shown that $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with $e_0 = 1/\sqrt{\mathrm{vol}(U)}$. Inserting (22) into (20)–(21), we get

$$\frac{d}{dt} a_k(t,\nu) + (\frac{1}{\nu^3} + \lambda_k \nu^3) a_k(t,\nu) = \frac{\nu e^{-\nu/M}}{M^2} \alpha_k(t)\,, \tag{23}$$

$$R \frac{d}{dt} \alpha_k(t) + \alpha_k(t) = \int_{\nu=0}^{\infty} \frac{a_k(t,\nu)}{\nu^3}\, d\nu\,, \tag{24}$$

$$R\alpha_0(t) + \int_{\nu=0}^{\infty} a_0(t,\nu) d\nu = 0\,.$$

Note that the last equation, which follows from energy conservation, (18)–(19), and the orthogonality property of the e_k, does not constrain the coefficients α_k, a_k for $k \geq 1$. Following the proof of Lemma 1, we can show that solutions of (23)–(24) satisfy $|a_k(t,\nu)| \leq \nu^4 M^{-2} e^{-\nu/M} B$ and $|\alpha_k(t)| < B$ for $t > 0$, if the conditions hold at $t = 0$.

To analyze the solutions of (23)–(24) we need the following result.

Lemma 3 *Let $M, R > 0$. For every $\lambda > 0$ there exist a $\sigma = \sigma(\lambda) > 0$ such that*

$$\int_{\nu=0}^{\infty} \frac{\nu e^{-\nu/M}}{M^2(1 + \lambda\nu^6 - \sigma\nu^3)}\, d\nu = 1 - R\sigma\,.$$

The function $\sigma(\lambda)$ is an increasing function of λ and there is an $\eta > 0$ such that $1 + \lambda\nu^6 - \sigma(\lambda)\nu^3 > \eta^2$ for all $\lambda > 0$, $\nu > 0$.

Proof Since $1+\lambda\nu^6-\sigma\nu^3 = 1-\sigma^2/(4\lambda)+\lambda[\nu^3-\sigma/(2\lambda)]^2$ we see that the integral – call it $f(\sigma)$ – is positive and well defined for $\sigma < 2\sqrt{\lambda}$. It is a smooth increasing function of σ and becomes unbounded as $\sigma \to 2\sqrt{\lambda}$. Using the property that $1-R\sigma$ decreases with σ and the inequality $0 < f(0) < 1$, we can find a $\sigma = \sigma(\lambda) > 0$ such that $f(\sigma) = 1 - R\sigma$. Note that $\sigma < \min(1/R, 2\sqrt{\lambda})$. Differentiating the identity in Lemma 3 with respect to λ we obtain

$$\frac{d}{d\lambda}\sigma(\lambda) = \int_{\nu=0}^{\infty} \frac{\nu^7 e^{-\nu/M} \quad d\nu}{M^2(1+\lambda\nu^6-\sigma\nu^3)^2} \Bigg/ \left(R + \int_{\nu=0}^{\infty} \frac{\nu^4 e^{-\nu/M} \quad d\nu}{M^2(1+\lambda\nu^6-\sigma\nu^3)^2}\right)$$

and $(d^2/d\lambda^2)\sigma(\lambda) < 0$. This implies that $\sigma(\lambda)$ is an increasing function of λ and concave down. If λ is small and positive, we have

$$\sigma(\lambda) = 0 + \frac{5040M^6}{R+24M^3}\lambda - \cdots .$$

Let $g(\lambda) = 1 - \sigma^2(\lambda)/(4\lambda)$ with $g(0) = 1$. Then, $g(\lambda) > 0$ for all $\lambda > 0$. Since $\sigma^2(\lambda)/(4\lambda) \to 0$ as $\lambda \to 0$ and $g(\lambda) \geq 3/4$ when $\lambda \geq 1/R^2$, we see that $g(\lambda)$ is continuous at $\lambda = 0$ and conclude that $g(\lambda) > \eta^2$ for all $\lambda \geq 0$. But $1+\lambda\nu^6-\sigma(\lambda)\nu^3 \geq g(\lambda)$. This completes the proof.

Theorem 2 *Let $a_k(t,\nu)$, $\alpha_k(t)$ be solutions of (23)–(24) satisfying*

$$|a_k(0,\nu)| \leq \frac{\nu^4 e^{-\nu/M}}{M^2} b_k\,, \qquad |\alpha_k(0)| \leq b_k\,,$$

for $k = 0, 1, \ldots$ and $\nu > 0$. Let $\sigma > 0$ satisfy the non-linear equation in Lemma 3 with $\lambda = \lambda_1$ where λ_1 is the smallest non-zero eigenvalue of $\Delta e(x) + \lambda e(x) = 0$, $x \in U$ with $(\partial/\partial n)e(x) = 0$ for $x \in \partial U$. If $b_0 = 0$ and $\sum_{k=1}^{\infty} b_k^2 < \infty$, there is a $C > 0$ such that for $t \geq 0$

$$\|u'(x,t,\nu)\|_{L^2(U)} \leq \frac{\nu^4 e^{-\nu/M}}{M^2} C e^{-0.9\sigma t}\,, \qquad \|T'(x,t)\|_{L^2(U)} \leq C e^{-0.9\sigma t}\,.$$

Remarks (1) The theorem asserts that if u', T' initially consist of only spatially varying components, i.e., $a_0(0,\nu) = \alpha(0) = 0$, the a_k, α_k modes for $k \geq 1$ decay to zero exponentially. Thus, for such perturbations about the stationary solution, the latter is an attractor. (2) If $a_0(0,\nu)$, $\alpha(0) \neq 0$, it is possible to show that $a_0(t,\nu)$ and $\alpha_0(t)$ tend to zero. However, the decay is very slow and requires a different proof. We discuss this in section 5.

Proof To prove Theorem 2, we derive bounds for a_k, α_k as functions of time. One might think that the higher Fourier coefficients go to zero rapidly, but this is not the case. The relaxation to equilibrium depends on R and the size of the domain, and can be slow for large R and large domains.

Let $k \geq 1$ be fixed and replace a_k, α_k, λ_k in (23)-(24) by a, α, λ. We will show that a particular linear combination of a and α goes to zero. Let $\sigma > 0$ be determined as in Lemma 3. If we subtract σa from both sides of (23), divide throughout by $1 + \lambda\nu^6 - \sigma\nu^3$, integrate for $\nu > 0$, and apply Lemma 3, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\nu=0}^{\infty} \frac{a(t,\nu)\,d\nu}{1+\lambda\nu^6-\sigma\nu^3} + \int_{\nu=0}^{\infty} \frac{a(t,\nu)\,d\nu}{\nu^3} \\
&\quad = \int_{\nu=0}^{\infty} \frac{(\nu/M^2)\,e^{-\nu/M}\,d\nu}{1+\lambda\nu^6-\sigma\nu^3}\,\alpha(t) - \sigma\int_{\nu=0}^{\infty} \frac{a(t,\nu)\,d\nu}{1+\lambda\nu^6-\sigma\nu^3} \\
&\quad = (1-R\sigma)\alpha(t) - \sigma\int_{\nu=0}^{\infty} \frac{a(t,\nu)\,d\nu}{1+\lambda\nu^6-\sigma\nu^3}\,.
\end{aligned}$$

Combining this result with (24) leads to

$$\frac{d}{dt}\left[R\alpha(t) + \int_{\nu=0}^{\infty} \frac{a(t,\nu)\,d\nu}{1+\lambda\nu^6-\sigma\nu^3}\right] = -\sigma\left[R\alpha(t) + \int_{\nu=0}^{\infty} \frac{a(t,\nu)\,d\nu}{1+\lambda\nu^6-\sigma\nu^3}\right]\,.$$

We can solve this differential equation and get

$$R\alpha(t) + \int_{\nu=0}^{\infty} \frac{a(t,\nu)\,d\nu}{1+\lambda\nu^6-\sigma\nu^3} = C_1 e^{-\sigma t}\,. \tag{25}$$

where C_1 is the initial value of the left hand side.

Our next goal is to derive a differential inequality for

$$A(t) = \int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M} a^2(t,\nu)\,d\nu}{\nu(1+\lambda\nu^6-\sigma\nu^3)}\,. \tag{26}$$

This time we multiply both sides of (23) by $2\nu^{-1}M^2e^{\nu/M}a/(1+\lambda\nu^6-\sigma\nu^3)$, integrate for $\nu > 0$, and obtain

$$\begin{aligned}
&\frac{d}{dt}A(t) + 2\int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M} a^2(t,\nu)}{\nu^4}\,d\nu \\
&\quad = 2\int_{\nu=0}^{\infty} \frac{a(t,\nu)}{1+\lambda\nu^6-\sigma\nu^3}\,d\nu\,\alpha(t)\; - 2\sigma A(t)\,.
\end{aligned} \tag{27}$$

Using (25) and completing the square we write the penultimate term as

$$\begin{aligned}
&2\int_{\nu=0}^{\infty} \frac{a(t,\nu)}{1+\lambda\nu^6-\sigma\nu^3}\,d\nu\,\alpha(t) \\
&\quad = -\frac{2}{R}\left(\int_{\nu=0}^{\infty} \frac{a(t,\nu)}{1+\lambda\nu^6-\sigma\nu^3}\,d\nu\; - \frac{1}{2}C_1 e^{-\sigma t}\right)^2 + \frac{C_1^2}{2R}e^{-2\sigma t}\,.
\end{aligned} \tag{28}$$

Inserting (26) and (28) in (27) yields

$$\begin{aligned}
&\frac{d}{dt}A(t) + 2\sigma A(t) + 2\int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M} a^2(t,\nu)}{\nu^4}\,d\nu \\
&\quad + \frac{2}{R}\left(\int_{\nu=0}^{\infty} \frac{a(t,\nu)}{1+\lambda\nu^6-\sigma\nu^3}\,d\nu\; - \frac{1}{2}C_1 e^{-\sigma t}\right)^2 = \frac{C_1^2}{2R}e^{-2\sigma t}\,.
\end{aligned}$$

Neglecting the integral terms gives a differential inequality for $A(t)$ with the solution

$$A(t) \le \left(A(0) + \frac{C_1^2}{2R}\, t \right) e^{-2\sigma t}\,.$$

In the arguments below it is inconvenient to carry the term proportional to t. However, as $te^{-\sigma t/5} \le 5/(\sigma e)$ for $t \ge 0$ we can use the weaker result

$$A(t) \le \left(A(0) + \frac{C_1^2}{R\sigma} \right) e^{-1.8\sigma t}\,. \tag{29}$$

Our next task is to estimate $\alpha(t)$ and $a(t,\nu)$. Using Cauchy-Schwarz in (25) we see that

$$\begin{aligned} |\alpha(t)| &= \frac{1}{R} \left| C_1 e^{-\sigma t} - \int_{\nu=0}^{\infty} \frac{a(t,\nu)}{1+\lambda\nu^6 - \sigma\nu^3}\, d\nu \right| \\ &\le \frac{1}{R} \left[|C_1| e^{-\sigma t} + \left(\int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M} a^2(t,\nu)}{\nu(1+\lambda\nu^6 - \sigma\nu^3)}\, d\nu \right)^{1/2} \right. \\ &\qquad \left. \times \left(\int_{\nu=0}^{\infty} \frac{\nu e^{-\nu/M}}{M^2(1+\lambda\nu^6 - \sigma\nu^3)}\, d\nu \right)^{1/2} \right]\,. \end{aligned}$$

In the product, the first integral is $A(t)$, while the second is $1 - R\sigma$, see (26) and Lemma 3. Since $R\sigma < 1$ and $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ we find from (29) that

$$\begin{aligned} |\alpha(t)| &\le \frac{1}{R} \left[|C_1| e^{-\sigma t} + \left(A(0) + \frac{C_1^2}{R\sigma} \right)^{1/2} e^{-0.9\sigma t} (1 - R\sigma)^{1/2} \right] \\ &\le \frac{1}{R} \left[\sqrt{A(0)} + \frac{2|C_1|}{\sqrt{R\sigma}} \right] e^{-0.9\sigma t} \\ &= C_2 e^{-0.9\sigma t}\,. \end{aligned} \tag{30}$$

To estimate $a(t,\nu)$ we rewrite (23) as

$$a(t,\nu) = e^{-(\nu^{-3}+\lambda\nu^3)t} a(0,\nu) + \int_{s=0}^{t} \frac{\nu e^{-\nu/M}}{M^2} e^{-(\nu^{-3}+\lambda\nu^3)(t-s)} \alpha(s) ds\,.$$

Using (30) and integrating with respect to s, we obtain

$$|a(t,\nu)| \le e^{-(\nu^{-3}+\lambda\nu^3)t} |a(0,\nu)| + \frac{\nu e^{-\nu/M}}{M^2}\, C_2\, \frac{e^{-0.9\sigma t} - e^{-(\nu^{-3}+\lambda\nu^3)t}}{\nu^{-3} + \lambda\nu^3 - 0.9\sigma}\,.$$

Since $1 + \lambda\nu^6 - \sigma\nu^3 > \eta^2$, see Lemma 3, it follows that $-(\nu^{-3} + \lambda\nu^3) < -0.9\sigma$ and

$$\begin{aligned} |a(t,\nu)| &\le \frac{\nu^4 e^{-\nu/M}}{M^2} \left(b_k + \frac{C_2}{\eta^2} \right) e^{-0.9\sigma t} \\ &= \frac{\nu^4 e^{-\nu/M}}{M^2}\, C_3\, e^{-0.9\sigma t}\,. \end{aligned} \tag{31}$$

Next, we derive bounds for $A(0)$, C_1, C_2, and C_3. After setting $t = 0$ in (26) and recalling the assumption, $|a(0,\nu)| \le \nu^4 M^{-2} e^{-\nu/M} b_k$, it follows from Lemma 3 that

$$\begin{aligned} |A(0)| &\le \int_{\nu=0}^{\infty} \frac{\nu^7\, e^{-\nu/M} b_k^2}{M^2(1+\lambda\nu^6-\sigma\nu^3)}\, d\nu \\ &\le \frac{7!\, M^6\, b_k^2}{\eta^2}\,. \end{aligned}$$

Similar arguments applied to (25) give

$$\begin{aligned} |C_1| &\le R|\alpha(0)| + \int_{\nu=0}^{\infty} \frac{|a(0,\nu)|}{1+\lambda\nu^6-\sigma\nu^3}\, d\nu \\ &\le \left(R + \frac{24\, M^3}{\eta^2}\right) b_k \end{aligned}$$

Combining these bounds, we conclude from (30) and (31) that

$$\begin{aligned} C_2 &\le \frac{1}{R}\left[\frac{71M^3}{\eta} + \frac{2}{\sqrt{R\sigma}}\left(R + \frac{24\, M^3}{\eta^2}\right)\right] b_k \quad \text{and} \\ C_3 &\le \left(1 + \frac{1}{R\eta^2}\left[\frac{71M^3}{\eta} + \frac{2}{\sqrt{R\sigma}}\left(R + \frac{24\, M^3}{\eta^2}\right)\right]\right) b_k\,. \end{aligned} \tag{32}$$

So far we have assumed that k is fixed. However, note that M, R, and η are independent of k. Since $\lambda_1 \le \lambda_2 \le \cdots$ and $\sigma(\lambda)$ is an increasing function of λ we see that $\sigma(\lambda_1) \le \sigma(\lambda_k)$ for $k \ge 1$. Thus, if we replace σ by $\sigma(\lambda_1)$ in the bounds for C_2, C_3 we find that $C_2,\, C_3 \le C_4\, b_k$, where C_4 is independent of k. Observe also that the slowest decay in (30), (31) occurs for $\sigma = \sigma(\lambda_1)$ where λ_1 is the eigenvalue of the lowest, spatially varying eigenmode.

Finally we must reassemble the Fourier series for u', T'. Since $\sum_{k=1}^{\infty} b_k^2 < \infty$ it follows from (22), (31), and (32) that

$$\begin{aligned} \|u'(x,t,\nu)\|_{L^2(U)} &= \|\sum_{k=1}^{\infty} a_k(t,\nu)\, e_k(x)\|_{L^2(U)} \\ &= \left(\sum_{k=1}^{\infty} a_k^2(t,\nu)\right)^{1/2} \\ &\le \frac{\nu^4 e^{-\nu/M}}{M^2}\, C\, e^{-0.9\sigma t}\,, \end{aligned}$$

where $C = C_4\, \left(\sum b_k^2\right)^{1/2}$. We can use the same arguments for T'. This completes the proof.

5 Asymptotic stability

In section 4, we linearized (10)–(12) around a stationary solution and showed that the solution of the linearized equations tends to zero exponentially. We therefore expect

that for every ν the solutions of the original equations will be approximately constant as times increases. In this section we consider a class of functions that are independent of x and show that they tend to a stationary solution. The stationary solution is therefore asymptotically stable for this class of initial data.

Theorem 3 *Let $u(t,\nu)$, $T(t)$ be independent of x, satisfy (10)–(11) and the bounds in Lemma 1. If*

$$R\,T(0) + \int_{\nu=0}^{\infty} u(0,\nu)\,d\nu \;=\; R\,M + 6M^4\,, \tag{33}$$

and

$$\left(\int_{\nu=0}^{\infty} \frac{M^2\,e^{\nu/M}}{\nu}\,|u(0,\nu) - \nu^3 e^{-\nu/M}|^2\,d\nu\right)^{1/2} < \frac{RM}{7} \tag{34}$$

then,

$$\begin{aligned} u(t,\nu) &\;\to\; \nu^3 e^{-\nu/M} && as\ t\to\infty,\ \nu>0\ fixed \\ T(t) &\;\to\; M && as\ t\to\infty\,. \end{aligned}$$

Proof As in the proof of Theorem 2 we set $u' = u - \nu^3 e^{-\nu/M}$ and $T' = T - M$, but now we do not assume that u', T' are small. It follows from (10)–(11) and the constraint (33) that

$$\frac{\partial}{\partial t}u'(t,\nu) + \frac{1}{\nu^3}u'(t,\nu) = e^{-\nu/[M+T'(t)]} - e^{-\nu/M}\,, \tag{35}$$

$$R\frac{d}{dt}T'(t) + T'(t) = \int_{\nu=0}^{\infty} \frac{u'}{\nu^3}\,d\nu\,, \tag{36}$$

$$R\,T'(t) + \int_{\nu=0}^{\infty} u'(t,\nu)\,d\nu = 0\,. \tag{37}$$

Equation (37) follows from the energy conservation property. We proceed as in the previous proof. This time we derive a differential inequality for

$$A(t) = \int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M}}{\nu}|u'(t,\nu)|^2\,d\nu\,, \tag{38}$$

and show that $A(t)$ decays. The function $A(t)$ defined in (38) is the analogue of the one defined in (26), except now we are considering the $\lambda = 0$ case, which according to Lemma 3 corresponds to $\sigma = 0$. Note that (34) and (38) imply that

$$0 < A(0) < (RM/7)^2\,. \tag{39}$$

Using Cauchy-Schwarz in (37) yields

$$\begin{aligned} R|T'(t)| &\leq \int_{\nu=0}^{\infty} |u'(t,\nu)|\, d\nu \\ &\leq \left(\int_{\nu=0}^{\infty} \frac{\nu e^{-\nu/M}}{M^2}\, d\nu\right)^{1/2} \left(\int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M}}{\nu} |u'(t,\nu)|^2\, d\nu\right)^{1/2} \\ &= 1 \cdot \sqrt{A(t)}\,. \end{aligned} \tag{40}$$

To analyze (35), we expand the right hand side using Taylor's formula,

$$\begin{aligned} e^{-\nu/(M+T')} - e^{-\nu/M} &= \frac{\nu e^{-\nu/M}}{M^2}\, T' \\ &+ \frac{1}{2}\, e^{-\nu/(M+\theta T')} \left[\frac{\nu^2}{(M+\theta T')^4} - \frac{2\nu}{(M+\theta T')^3}\right] (T')^2\,, \end{aligned} \tag{41}$$

where $0 < \theta < 1$.

We now prove that $|T'(t)| < M/7$ for all t and do this by contradiction. Equations (40), (38), and (39) show that at least for small t,

$$|T'(t)| \;\leq\; R^{-1}\sqrt{A(t)} < M/7\,.$$

Let $t = t_2$ be the first time for which the inequality fails; thus, $|T'(t_2)| = M/7$. Since $|T'(t)| \leq M/7$ for $t \leq t_2$, we see that the absolute value of the term multiplying $(T')^2$ in (41) is bounded by

$$N(\nu) = \frac{1}{2}\, e^{-7\nu/(8M)} \left[\frac{\nu^2}{(6M/7)^4} + \frac{2\nu}{(6M/7)^3}\right].$$

Hence, there exists a $\gamma(\nu)$ with $|\gamma| \leq 1$ such that

$$e^{-\nu/(M+T')} - e^{-\nu/M} = \frac{\nu e^{-\nu/M}}{M^2}\, T' + \gamma\, N(\nu)\, (T')^2\ .$$

Inserting this expression in (35), multiplying both sides by $\nu^{-1} M^2 e^{\nu/M} u'(t,\nu)$ and integrating over $\nu > 0$, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M}}{\nu} |u'|^2\, d\nu + \int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M}}{\nu^4} |u'|^2\, d\nu \\ &\quad = \left(\int_{\nu=0}^{\infty} u' d\nu\right) T' + \left(\int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M}}{\nu}\, u'\, \gamma\, N(\nu)\, d\nu\right) (T')^2\,. \end{aligned} \tag{42}$$

Note that the first term is $(1/2)dA/dt$. Let E_1, E_2 denote the terms on the right hand side of (42). It is clear from (37) that $E_1 = -R\,[\,T'(t)\,]^2$. To estimate E_2, we recall that $|\gamma| \leq 1$, apply Cauchy-Schwarz, and get

$$\begin{aligned} |E_2| \leq &\left(\int_{\nu=0}^{\infty} \frac{M^2 e^{\nu/M}}{\nu} |u'(t,\nu)|^2\, d\nu\right)^{1/2} \\ &\times \left(\int_{\nu=0}^{\infty} \left[\frac{M\, e^{\nu/(2M)}}{\sqrt{\nu}}\, N(\nu)\right]^2 d\nu\right)^{1/2} (T')^2\,. \end{aligned}$$

The first integral is $A(t)$, see (38), while an elementary, but tedious calculation shows that the second integral is less than $36/M^2$. Thus, $|E_2| \leq (6/M)\sqrt{A(t)}$. Inserting the expressions for E_1, E_2 in (42), we obtain the desired differential inequality,

$$\frac{1}{2}\frac{d}{dt}A(t) + \int_{\nu=0}^{\infty} \frac{M^2\, e^{\nu/M}}{\nu^4}|u'(t,\nu)|^2\, d\nu \;+\; \left(R - \frac{6}{M}\sqrt{A(t)}\right) |\, T'(t)\,|^2 \leq 0\,. \quad (43)$$

We note that for the linearized problem discussed in the previous section, we would have arrived at the same inequality, but without the term $(6/M)\sqrt{A(t)}$ and with u', T' replaced by a_0, α_0.

Equations (39) and (43) show that $A(t)$ is a decreasing function of t for $t \leq t_2$. Thus, $A(t)^2 \leq A(0)^2 < RM/7$. However, (40) shows that $|T'(t)| \leq R^{-1}\sqrt{A(t)} < M/7$ for $0 \leq t \leq t_2$ which contradicts the definition of t_2. Thus, the bound,

$$|T'(t)| \leq M/7\,, \quad (44)$$

holds for all time.

We now show that $T'(t)$ decays to zero. If in (43) we ignore the middle term, note that $R - 6\sqrt{A(t)}/M > R/7$, and integrate we obtain,

$$A(t) + 2\int_{\tau=0}^{t} \frac{R}{7}\,|T'(\tau)|^2\, d\tau \leq A(0)\,.$$

The function $|T'(t)|^2$ is therefore integrable on $(0,\infty)$. However, this is insufficient for our purposes since we wish to show that $T'(t) \to 0$ as $t \to \infty$ (which implies $T \to M$.)

To prove $T' \to 0$, we first show that $|(d/dt)[T'(t)]^2|$ is bounded. Equation (44) leads to

$$\left|\frac{d(T')^2}{dt}\right| \leq 2\,|T'(t)|\,\left|\frac{dT'}{dt}\right| \leq \frac{2M}{7}\,\left|\frac{dT'}{dt}\right|\,. \quad (45)$$

Using (36), (44), and the definition of $u'(x,\nu)$, we obtain

$$\left|\frac{dT'}{dt}\right| \leq \frac{1}{R}\,\left[\,\frac{M}{7} + \left|\int \nu^{-3}\,(u - \nu^3 e^{-\nu/M})\, d\nu\right|\,\right]\,.$$

If T_2 is the upper bound defined in Lemma 1, the last integral is bounded by $\max(\,M,\,T_2 - M) < T_2$. Substituting these results into (45) yields,

$$|(d/dt)[T'(t)]^2| \leq (2M/7)\,(1/R)\,(T_2 + M/7) \doteq c\,.$$

Thus, the function $|T'(t)|^2$ is integrable and has a uniformly bounded derivative. Clearly, $|T'(t)|^2$, for large t, cannot stray too far from zero. In fact, for any $\epsilon > 0$, $|T'(t)|^2$ can only have a finite number of spikes above ϵ no less than $2\epsilon/c$ apart, because every neighborhood with radius ϵ/c of a spike contributes at least ϵ^2/c toward the area under the curve. Thus $|T'(t)|^2 < \epsilon$ for t sufficiently large.

To show that $u' \to 0$ when ν is fixed and $t \to \infty$, we solve (35) and get

$$\begin{aligned} u'(t,\nu) &= e^{-t/\nu^3} u'(0,\nu) + \int_{\tau=0}^{t} e^{-(t-\tau)/\nu^3} \left(e^{-\nu/[M+T'(\tau)]} - e^{-\nu/M} \right) d\tau \\ &\doteq E_3 + E_4 \,. \end{aligned}$$

Clearly $E_3 \to 0$ as $t \to \infty$. For E_4 we use L'Hospital. Since $e^{t/\nu^3} \to \infty$ and $T'(t) \to 0$ as $t \to \infty$ it follows that

$$\lim_{t\to\infty} E_4(t) = \lim_{t\to\infty} \frac{\int_{\tau=0}^{t} e^{\tau/\nu^3} \left(e^{-\nu/[M+T'(\tau)]} - e^{-\nu/M} \right) d\tau}{e^{t/\nu^3}}$$

$$= \lim_{t\to\infty} \frac{e^{-\nu/[M+T'(t)]} - e^{-\nu/M}}{(1/\nu^3)} = 0 \,.$$

Combining the results for E_3, E_4 completes the argument. Our proof is nonconstructive and gives no rate of convergence. However, numerical experiments indicate very slow convergence. This completes the proof.

6 Concluding remarks

We have derived a nondimensional form of the system that couples the multifrequency radiation diffusion equations to the matter energy balance equation. In deriving the nonlinear system, we strived for equations that preserve the salient properties of the underlying physics, yet be amenable to analysis. Our equations may be applied in multiple spatial dimensions and arbitrarily sized domains on which one may impose boundary conditions appropriate to parabolic systems.

Our nondimensional system conserves total energy and a unique stationary solution is eventually established. For homogeneous Neumann boundary conditions, the stationary solution is spatially constant. We have shown that the decay to equilibrium is constrained by an envelope whose extent depends on the initial conditions. Spatially varying perturbations about the stationary solution decay to zero exponentially. Spatially constant perturbations also decay. However, we are unable to estimate the speed of decay; it may be very slow.

Although our nondimensional system is a valid model for multifrequency radiation diffusion, it is instructive to recall how it differs from equations typically used in high energy density computer codes.

We assume the matter to be composed of a single material, of constant density, characterized by an ideal gas equation-of-state. For "real" materials, the internal energy is a nonlinear function of temperature [12], p.177. This implies a temperature dependence for the specific heat, $c_{\rm v} = c_{\rm v}(T)$. Recalling (7), we obtain the dependence $R = R(T)$ for our model.

If matter density were to vary with position, so would the opacities. This implies spatial variations for the diffusion coefficient ν^3 in (8), the coupling coefficient $1/\nu^3$ in (8)–(9), and the coefficient R in (9). For real materials, especially for those of high atomic number and/or at low temperatures, the opacity is a complicated, nonmonotonic function of frequency [12], p.246. Most notably, it features occasional spikes corresponding to the "bound–bound" transitions of the electrons.

Our model ignores the $1/\sqrt{T}$ dependence of the opacity. As we show below, including $1/\sqrt{T}$ leads to spatial variations of the diffusion and coupling coefficients. This complicates the analysis significantly. We are then unable to invoke Duhamel's principle since a general Green's function $G_\nu(x, y, t)$ does not exist.

We chose a Wien $\nu^3 e^{-\nu/T}$ rather than a Planck $\nu^3/(e^{\nu/T} - 1)$ distribution for radiation emission. The difference between the two is most pronounced for small ν yielding a smaller radiation source. However, since our absorption opacity ignores induced emission [12], multiplying our κ_ν by a Wien distribution gives the correct total radiation source, if we ignore the $1/\sqrt{T}$ dependence of the opacity. To see this, recall that to include induced emission, one multiplies κ_ν by the factor $(1 - e^{-y})$, where $y = h\nu/kT$. Then, if instead of (6), we use the Planck function, the exponential is replaced by the factor $1/(e^y - 1)$. Hence, if B_ν denotes the Planck function and κ'_ν the absorption coefficient corrected for stimulated emission, the $(e^y - 1)$ factors cancel and we obtain the same total radiation source,

$$c \int_0^\infty d\nu \, \kappa'_\nu \, B_\nu = \text{const}\, T \ .$$

Choosing a Wien rather than a Planck function brings an additional benefit when the system is discretized in the frequency direction in order to derive the "multigroup" equations. For this endeavor, one defines a spectral mesh

$$0 = \nu_0 < \nu_1 < \ldots < \nu_N \ .$$

If B_ν is the Wien function, integrals of the form

$$\int_{\nu_{j-1}}^{\nu_j} d\nu \, \nu^n \, B_\nu$$

can be done analytically. If B_ν is the Planck function, the integrals can only be approximated.

Lastly, we did not include a flux limiter in the diffusion coefficient. This leads to unphysically large photon speeds for our model, especially for those with high frequencies. To incorporate a flux limiter into our model, we note that (physically) the distance x that radiation can travel in a time interval t, is limited by $x \leq c\,t$. Recalling the nondimensional variables x' and t', the inequality reduces to $x' \leq \sqrt{3}\, t'$. Hence, if we introduce a Lund-Wilson type flux limiter, our nondimensional flux becomes

$$-\nu^3 \, \nabla u \to \frac{-1}{1/(\nu^3 \sqrt{T}) + |\nabla u|/(\sqrt{3}\, u)} \, \nabla u \doteq -D_\nu \, \nabla u \ . \tag{46}$$

Note that we have kept the dependence of κ_ν on $1/\sqrt{T}$.

We conclude the discussion of how to modify our model to account for some of the above effects. If we include the $\sqrt{T}$ dependence of the opacity, allow for a non-constant $c_{\rm v}$, write the emission function in the general form $B_\nu = B_0\,\nu^3\,g(\nu/T)$, but do not include effects of stimulated emission, the nondimensional equations become

$$\begin{aligned} \partial_t u &= \nabla\cdot D_\nu \nabla u + \left(\nu^3\,g(\nu/T) - u\right)/\sqrt{T}\,\nu^3\,, \\ R(T)\,\partial_t T &= -\frac{1}{\sqrt{T}}\left(\int_0^\infty d\nu\,\frac{1}{\nu^3}(\nu^3\,g(\nu/T) - u\,)\right)\,. \end{aligned}$$

where D_ν may be defined as in (46), with or without the flux limiter and/or the $\sqrt{T}$ factor, and $g(y)$ is chosen appropriate to either the Wien or Planck distribution.

The analysis described in this paper was motivated by an unexpected result from simulations, [10] and [11], which solve the multigroup system derived from (8)–(9). The papers simulate the relaxation to steady state of an initial condition in which the two fields, u and T, are wildly out of equilibrium.

In the problem of interest, the spatial domain is, $|x| < 1$. Initially, T is sharply peaked,

$$T|_{t=0} = \begin{cases} 6.4775\,, & \text{if } |x| < 0.04 \\ 0.0027\,, & \text{otherwise} \end{cases}\,.$$

Using a "specific heat" $R = 2.0$ yields an initial matter energy $\mathcal{E}_m(t)|_{t=0} = 1.0469$. The spectral radiation energy density u is initialized with a Wien profile characterized with a spatially constant "radiation temperature" $T_r = 0.0172$. Hence, the initial radiation energy $\mathcal{E}_r(0) = 1.0469\cdot 10^{-6}$. The conditions imply that initially 99% of the total energy $\mathcal{E}_m + \mathcal{E}_r$ is contained in the central region $|x| < 0.04$. The problem is designed so that the stationary solution consists of spatially uniform fields with the matter energy equal to 1, i.e., with $T = 0.25$.

In the simulation, the hot central region pumps a prodigious amount of energy into the high frequencies. (A result consistent with the Wien profile which peaks at $\nu/T = 3$.) The high frequencies are characterized by fast transport. Thus, this energy quickly diffuses away from the hot spot. However, the high frequency energy is slow to absorb since the coupling coefficient is $1/\nu^3$. In effect, the high frequency energy is trapped for a time proportional to ν^3. Because of the trapping, the ratio $\mathcal{E}_r(t)/\mathcal{E}_m(t)$ exhibits a surprising behavior. It quickly rises to approximately 0.6 (more than $10\times$ the equilibrium value), then gradually decays, albeit very slowly. That very slow relaxation to equilibrium is the subject of section 5. Durung the slow-relaxation phase, the fields have little spatial variation.

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