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UCRL-JRNL-202956

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D.H. Chambers, J. V. Candy, S. K. Lehman, J. S. Kallman, A. J. Poggio, A. W. Meyer

March 8, 2004

Submitted for publication in the Journal of the Acoustical Society of America

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D. H. Chambers, J. V. Candy, S. K. Lehman, J. S. Kallman, A. J. Poggio, and A. W. Meyer

Lawrence Livermore National Laboratory

Livermore, CA 94551-9900[*]

(Dated: June 8, 2004)

Abstract

It is known that focusing of an acoustic field by a time-reversal mirror (TRM) is equivalent to a spatio-temporal matched filter under conditions where the Green's function of the field satisfies reciprocity and is time invariant, *i.e.* the Green's function is independent of the choice of time origin. In this letter, it is shown that both reciprocity and time invariance can be replaced by a more general constraint on the Green's function that allows a TRM to implement the spatio-temporal matched filter even when conditions are time varying.

PACS numbers: 43.60.Gk

[*]Electronic address: `chambers2@llnl.gov`

For over a decade, time-reversal focusing has been described as an implementation of a spatio-temporal matched filter. This was first derived by Fink[1] who showed that the fields produced by each element in a time-reversal array added coherently at the focus. Dorme[2] showed how this can be realized on reception of a signal by an array. Recently, Tanter *et al.*[3] proved that time-reversal produced the optimal spatial matched filter. This work established that time invariance and reciprocity are *sufficient* for time reversal focusing to be the optimal spatio-temporal matched filter. (If $G(\mathbf{x}_2, t_2; \mathbf{x}_1, t_1)$ is the Green's function for the acoustic system, then time invariance implies $G = G(\mathbf{x}_2, \mathbf{x}_1, t_2 - t_1)$, and reciprocity implies $G(\mathbf{x}_2, \mathbf{x}_1, t_2 - t_1) = G(\mathbf{x}_1, \mathbf{x}_2, t_2 - t_1)$ [1].) A related but still unanswered question is whether both time invariance and reciprocity are *necessary* for a time reversal array to produce the optimal spatio-temporal matched filter. Though this would seem physically intuitive, we will show mathematically that time invariance and reciprocity are not strictly necessary for time-reversal focusing to be the spatio-temporal matched filter. We first derive the spatio-temporal matched filter for a general (time-variant) wave system using a method similar to that of Tanter[3] and Cox[4]. This is compared with the response of a time-reversal array system to a point source in the medium, which leads to a necessary condition on the Green's function for time reversal focusing to be equivalent to the spatio-temporal matched filter. This condition is naturally satisfied for Green's function of time-invariant wave systems that obey reciprocity. However, it is also possible to formulate Green's functions that satisfy the condition but do not satisfy reciprocity or time-invariance.

In signal processing texts, the matched filter is typically derived for single channel time series (see [5, 6]). A signal $u(t)$ is input into a filter with impulse response $h(t)$, resulting in an output $y(t) = h(t) * u(t)$ ($*$ indicates convolution). The input is assumed to be a combination of signal $s(t)$ and additive white noise $w(t)$ ($u = s + w$). The filter $h(t)$ is chosen to maximize the output signal-to-noise ratio at a specified time T ($SNR(T)$). The SNR is given by

$$SNR(T) = \frac{\left| \int_0^T h(t') s(T - t')\, dt' \right|^2}{\sigma^2 \int_0^T |h(t')|^2\, dt'}, \tag{1}$$

where $E\{w(t)w^*(t + \tau)\} = \sigma^2 \delta(\tau)$. The filter that maximizes $SNR(T)$ is $h(t) = s(T - t)$, which is the *matched* filter. Since the denominator of the SNR can be interpreted as the total energy in the filter response in the interval $0 < t < T$ [6], the matched filter is also the $h(t)$ that maximizes the numerator given the constraint of constant energy.

For multi-channel signal processing, such as beamforming, the output $y(t)$ is given by

$$y(t) = \sum_{n=1}^{N} \int_0^t h_n(t-t') x_n(t) \, dt' \; , \tag{2}$$

where $\{x_n(t) : n = 1, 2, ..., N\}$ is the set of input time series, and $\{h_n(t) : n = 1, 2, ..., N\}$ is the set of filters. In this case the SNR at time T is

$$SNR(T) = \frac{\left| \sum_{n=1}^{N} \int_0^T h_n(t') s(T-t') \, dt' \right|^2}{\sigma^2 \sum_{n=1}^{N} \int_0^T |h_n(t')|^2 \, dt'} \; , \tag{3}$$

where $E\{w_m(t) w_n^*(t+\tau)\} = \sigma^2 \delta_{mn} \delta(\tau)$. Maximizing the SNR leads to the MMSE beamformer described in Van Trees [7]. Again, this can be interpreted as maximizing the numerator given the constraint of constant total energy for the filters.

Consider now an array of N acoustic elements (point sources) at positions $\mathbf{a}_n$ ($n = 1, 2, ..., N$) radiating into a medium. Given the set of excitations $\{E_n(t) : n = 1, 2, ..., N\}$, the resulting acoustic field is

$$\psi(\mathbf{x}, t) = \sum_{n=1}^{N} \int_0^t G(\mathbf{x}, t; \mathbf{a}_n, t') E_n(t') \, dt' \; , \tag{4}$$

where the (complex) Green's function $G(\mathbf{x}, t; \mathbf{x}_s, t_s)$ specifies the response at spatial location $\mathbf{x}$ and time t to an impulse at $\mathbf{x}_s$ and time t_s ($G = 0$ for $t < t_s$ from causality). This expression is similar to equation (2) for the beamformer if we identify the field $\psi(\mathbf{x}, t)$ with the output $y(t)$, the excitations $E_n(t)$ with the filters $h_n(t)$, and the Green's functions $G(\mathbf{x}, t; \mathbf{a}_n, t')$ with the input data $x_n(t)$. We can define an equivalent signal-to-noise ratio $SNR(\mathbf{x}_0, T)$ as the ratio of $|\psi|^2$ at a given position $\mathbf{x}_0$ and time T to the total energy in the excitations in the interval $0 < t < T$. (This is the square of the functional used by Tanter, *et al.*[3]). The vector form of the Schwartz inequality, as found in Cox[4], is

$$\left| \sum_{n=1}^{N} \int f_n^*(t) g_n(t) \, dt \right|^2 \leq \left(\sum_{n=1}^{N} \int |f_n(t)|^2 \, dt \right) \left(\sum_{n=1}^{N} \int |g_n(t)|^2 \, dt \right) \; , \tag{5}$$

where the equality holds when $g_n(t) = k f_n(t)$ and k is a constant independent of n. We use this to bound $SNR(\mathbf{x}_0,T)$:

$$SNR(\mathbf{x}_0,T) = \frac{|\psi(\mathbf{x}_0,T)|^2}{\sum_{n=0}^{N}\int_0^T |E_n(t)|^2\,dt}$$
$$= \frac{\left|\sum_{n=1}^{N}\int_0^T G(\mathbf{x}_0,T;\mathbf{a}_n,t)E_n(t)\,dt\right|^2}{\sum_{n=0}^{N}\int_0^T |E_n(t)|^2\,dt}$$
$$\leq \sum_{n=1}^{N}\int_0^T |G(\mathbf{x}_0,T;\mathbf{a}_n,t)|^2\,dt\;, \tag{6}$$

with equality when $E_n(t) = G^*(\mathbf{x}_0,T;\mathbf{a}_n,t)$. The spatio-temporal matched filter maximizes $SNR(\mathbf{x}_0,T)$, which occurs when the equality condition is met. The resulting field is

$$\psi_{MF}(\mathbf{x},t) = \sum_{n=1}^{N}\int_0^t G(\mathbf{x},t;\mathbf{a}_n,t')\,G^*(\mathbf{x}_0,T;\mathbf{a}_n,t')\,dt'\;. \tag{7}$$

To implement the matched filter for a time-variant system, the Green's function must be known *a priori*. It cannot be obtained from direct experimental measurement because the conditions for which a measured Green's function would be valid would have changed by the time the measurement is completed. An exception might be a time-varying system that is periodic, *i.e.* $G(\mathbf{x},t;\mathbf{x}',t'+T') = G(\mathbf{x},t;\mathbf{x}',t')$ for some T'. If the period is known, a measured Green's function could be stored for use when conditions are repeated.

We compare the matched filter result to that obtained by assuming the array acts as a time reversal mirror. Suppose a source at position $\mathbf{x}_0$ emits an impulse at time $t=0$. The field sampled at each array element would be $\psi(\mathbf{a}_n,t) = G(\mathbf{a}_n,t;\mathbf{x}_0,0)$. If these are recorded over the period $0<t<T$, time reversed, and emitted by the array, the resulting field would be

$$\psi_{TR1}(\mathbf{x},t+T) = \sum_{n=1}^{N}\int_0^t G(\mathbf{x},t+T;\mathbf{a}_n,t'+T)\,G(\mathbf{a}_n,T-t';\mathbf{x}_0,0)\,dt'\;,\; t>0\;. \tag{8}$$

This could be realized, in principle, for a time-variant wave system because it does not require prior knowledge of the Green's function. However, if one has a model of the Green's function or the conditions are repeatable, the time-reversed field could be emitted simultaneously with the source pulse. The resulting field would be

$$\psi_{TR2}(\mathbf{x},t) = \sum_{n=1}^{N}\int_0^t G(\mathbf{x},t;\mathbf{a}_n,t')\,G(\mathbf{a}_n,T-t';\mathbf{x}_0,0)\,dt'\;,\; t>0\;. \tag{9}$$

This second result is equal to the field produced by the matched filter (equation (7)) if the Green's function satisfies the condition

$$G(\mathbf{a}_n,T-t;\mathbf{x}_0,0) = G^*(\mathbf{x}_0,T;\mathbf{a}_n,t)\;,\; 0<t<T\;. \tag{10}$$

This states that the time-reversed response of the system at $\mathbf{a}_n$ to an impulse emitted at position $\mathbf{x}_0$ and time 0 is equal to the complex conjugate of the response at $\mathbf{x}_0$ and time T to an impulse emitted at position $\mathbf{a}_n$ and time t. For a time invariant system the condition reduces to

$$G(\mathbf{a}_n;\mathbf{x}_0;t-t') = G^*(\mathbf{x}_0;\mathbf{a}_n,t-t'),\ t>t'\ , \tag{11}$$

which is a generalization of reciprocity to a wave system with a complex Green's function. For a real Green's function, the condition agrees with the statements of reciprocity used by Fink [1] and others [2, 3].

In summary, we have derived a mathematical condition (Eqn. (10)) on the Green's function such that time-reversal and matched filtering are equivalent for a general time-variant acoustic system. It is not difficult to find functions that obey this condition, *e g.*

$$G(\mathbf{x}_1,t_1;\mathbf{x}_2,t_2) = f(\mathbf{x}_1)f^*(\mathbf{x}_2)e^{i\omega(t_1+t_2-T)}\ . \tag{12}$$

This generalizes the earlier analysis by Fink and others [1–3] to systems that are not time-invariant. Actual implementation with a time-reversal array system would be possible only in cases where conditions repeat periodically. This could always be accomplished in a laboratory environment. Examples of time-variant systems that obey Eqn. (10) will be the subject of future study.

Acknowledgments

This work was performed under the auspices of the U. S. Department of Energy by the University of California, Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.

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