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March 17, 2004

17th ASCE Engineering Mechanics Conference
Newark, DE, United States
June 13, 2004 through June 16, 2004

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COMPARISON OF TWO UP-SCALING METHODS IN POROELASTICITY AND ITS GENERALIZATIONS

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ABSTRACT

Two methods of up-scaling coupled equations at the microscale to equations valid at the mesoscale and/or macroscale for fluid-saturated and partially saturated porous media are discussed, compared, and contrasted. The two methods are: (1) two-scale and multiscale homogenization, and (2) volume averaging. Both these methods have advantages for some applications and disadvantages for others. For example, homogenization methods can give formulas for coefficients in the up-scaled equations, whereas volume averaging methods give the form of the up-scaled equations but generally must be supplemented with physical arguments and/or data in order to determine the coefficients. Homogenization theory requires a great deal of mathematical insight from the user in order to choose appropriate scalings for use in the resulting power-law expansions, while volume averaging requires more physical insight to motivate the steps needed to find coefficients. Homogenization often is performed on periodic models, while volume averaging does not require any assumption of periodicity and can therefore be related very directly to laboratory and/or field measurements. Validity of the homogenization process is often limited to specific ranges of frequency – in order to justify the scaling hypotheses that must be made – and therefore cannot be used easily over wide ranges of frequency. However, volume averaging methods can quite easily be used for wide band data analysis.

Keywords: poroelasticity, homogenization, volume averaging, up-scaling

INTRODUCTION

The earth is typically probed with seismic waves in the range 1 – 100 Hz, with well-logging tools in the range 1 – 50 kHz, and samples of the earth in the laboratory from 200 – 1000 kHz. The pertinent wave speeds for water and typical solid earth materials like quartz are, respectively, 1.5 km/s and about 6.0 km/s. So the range of wavelengths of interest in the field can vary from as much as 60 to 6000 m in the field to as little as 1.5 to 7.5 mm in the laboratory. Clearly the main purpose of laboratory measurements of earth materials is generally to elucidate the physical mechanisms of wave propagation in the earth. But the differences in the pertinent length scales is so great that unusual care must be taken to perform proper interpretation of the results — taking into account all the inherent problems with up-scaling. In particular, since earth materials are notoriously heterogeneous, it is very important to have some means of studying the effects of these heterogeneities on waves. So up-scaling in earth sciences applications is often a critical issue for many important applications.

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The most common approach to dealing with earth heterogeneity for seismic waves is to assume the earth is homogeneous locally, but composed of many layers (Ewing et al. 1957; Brekhovskikh 1980). This approach can be useful for applications to large scale earth imaging and earthquake analysis. But in matters where fluids in the earth are important, such as oil and gas exploration, hydrology, etc., the elastic approximation is usually not good enough and must either be supplemented or replaced altogether with more appropriate choices of equations and analysis methods. Early examples of such analyses include Biot (1941), Frenkel (1944), Gassmann (1951), Biot (1956a), Biot (1956b), Biot (1962) — all works which then provided a strong foundation for modern poroelastic analysis. Sometimes viscoelastic analysis (Budiansky and O’Connell 1976; O’Connell and Budiansky 1977) is used instead of poroelastic analysis, but there are both laboratory and field data that cannot be adequately explained by viscoelastic analysis, suggesting that the full poroelastic approach is essential in such cases.

Our goal then will be to give a brief accounting of just two of the most important methods used to do up-scaling in poroelasticity and also multi-scale poroelasticity. The methods we consider are: (1) two-scale and multiscale homogenization (Burrige and Keller 1981; Auriault and Boutin 1994; Auriault and Royer 2002; Auriault 2002), and (2) volume averaging (Pride et al. 1992; Pride and Berryman 1998; Berryman and Pride 1998; Whitaker 1999; Whitaker 2002; Wood et al. 2003). Both these methods have advantages for some applications and disadvantages for others. In a longer review, we would also include (3) effective medium theory and (4) mixture theory. But we must limit discussion here just to homogenization and volume averaging.

HOMOGENIZATION THEORY

A two-space method of homogenization leading to equations having the form of Biot’s equations has been presented by Burrige and Keller (1981). This method has been developed by various authors including Bensoussan et al. (1978), Keller (1977), and Sanchez-Palencia (1980). The method requires that the microscale of the heterogeneous porous medium is much smaller than the macroscale of most interest. The method is systematic, leading to equations at the macroscale from an analysis of the microscale behavior, which for the present problem involves assuming the the solid components obey linearized equations of elasticity, while the fluid components obey linearized Navier-Stokes equations. Burrige and Keller (1981) show that there are actually two possible solutions to the problem. One solution is essentially that of Biot’s theory of wave propagation in poroelastic media. The other outcome is a set of viscoelastic equations [recall Budiansky and O’Connell (1976) and O’Connell and Budiansky (1977)]. The small quantity ϵ , being the ratio of the microscale size to the macroscale size, is used to characterize various scaling regimes. The difference leading to the two quite different results found by Burrige and Keller is that, when the scaled viscosity is treated as being of order ϵ^2 , they get the Biot-Gassmann equations, whereas when it is treated as order unity, they obtain equations of viscoelasticity instead. In the language of poroelasticity, the case leading to viscoelastic equations is what is normally termed “undrained,” meaning that the fluid does not have sufficient time for its pressure to equilibrate at the microscale throughout the macromedium on the time scales of interest. This failure to equilibrate can occur due to low fluid permeabilities, high viscosity, very high wave frequencies, or combinations of all these effects when present.

This approach involves assuming that any quantity Q can be treated as if it is a function of the two spatial scales \mathbf{x} and $\mathbf{y} = \mathbf{x}/\epsilon$. The macroscale is \mathbf{x} and the microscale is \mathbf{y} . Spatial

gradients ∇ of Q can then be usefully written as

$$\nabla Q(\mathbf{x}, \mathbf{y}) = \nabla Q(\mathbf{x}, \mathbf{x}/\epsilon) = \nabla_{\mathbf{x}} Q + \epsilon^{-1} \nabla_{\mathbf{y}} Q. \quad (1)$$

Thus, the scale separation can be explicitly and simply accounted for in such gradient equations. Furthermore, each quantity Q can also be treated as a function of ϵ , so that an asymptotic expansion of the form

$$Q(\mathbf{x}, \mathbf{y}, \epsilon) = Q_0(\mathbf{x}, \mathbf{y}) + \epsilon Q_1(\mathbf{x}, \mathbf{y}) + \frac{\epsilon^2}{2} Q_2(\mathbf{x}, \mathbf{y}) + O(\epsilon^2) \quad (2)$$

may be written. Combining (1) and (2) gives

$$\nabla Q = \epsilon^{-1} \nabla_{\mathbf{y}} Q_0(\mathbf{x}, \mathbf{y}) + [\nabla_{\mathbf{x}} Q_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} Q_1(\mathbf{x}, \mathbf{y})] + O(\epsilon), \quad (3)$$

a result which gets used repeatedly in the subsequent analysis. Furthermore, Eq. (3) already suggests the important result that, when ϵ is small — *i.e.*, tending to zero, it must generally be true that

$$\nabla_{\mathbf{y}} Q_0(\mathbf{x}, \mathbf{y}) = 0, \quad (4)$$

which is in fact a common result of this analysis.

If we let Ω_s be the domain occupied by solid, Ω_f the domain occupied by fluid, and ∂I_{sf} be the interface between solid and fluid, then the linearized equations for elasticity of the solid in Ω_s are

$$-\omega^2 \rho_s \mathbf{u}_s = \nabla \cdot \tau, \quad \text{where} \quad \tau = L \nabla \mathbf{u}_s, \quad (5)$$

the linearized equations of Navier-Stokes for the fluid are

$$i\omega \rho_f \mathbf{v}_f = \nabla \cdot \sigma_f \quad \text{where} \quad \sigma_f = -p_f I + \nu D \nabla \mathbf{v}_f \quad \text{and} \quad i\omega p_f = -\nabla \cdot \mathbf{v}_f / K_f. \quad (6)$$

The boundary conditions at the interfaces ∂I_{sf} are no slip: $\mathbf{v}_f = i\omega \mathbf{u}_s$, and continuity of normal stress: $\mathbf{n} \cdot \sigma_f = \mathbf{n} \cdot \tau$. The fluid and solid densities are ρ_f and ρ_s , respectively. The fluid viscosity is ν , and its bulk modulus is K_f . The stress tensors for fluid and solid are σ_f and τ , respectively, and p_f is the fluid pressure. L is the fourth rank elastic stiffness tensor, and D is the operator that produces the symmetrized deviatoric part of a second rank tensor.

We will use a notation slightly different from that of Burrige and Keller (1981) in order to facilitate the comparisons between these results and those of Biot. Space constraints will not permit us to follow the derivation of the equations further here. But one of the final macroscale results of the analysis is given by

$$-\omega^2 (\rho \mathbf{u}_0 + \rho_f \overline{\mathbf{w}}) = \nabla_{\mathbf{x}} \cdot (\overline{\tau}_0 - \phi p_0 I), \quad (7)$$

where $\rho = (1 - \phi) \rho_s + \phi \rho_f$, and ϕ is the porosity. The overbar indicates a volume average over the fast variable \mathbf{y} . The second macroscale result is

$$-\omega^2 [\rho_f \mathbf{u}_0 + \Gamma(\omega) \overline{\mathbf{w}}] = -\nabla_{\mathbf{x}} p_0. \quad (8)$$

where $\Gamma(\omega)$ is a viscodynamic operator. The theory also shows that the macroscale stress and fluid pressure are determined by

$$\overline{\tau}_0 - \phi p_0 I = J \nabla_{\mathbf{x}} \mathbf{u}_0 + C \nabla_{\mathbf{x}} \cdot \overline{\mathbf{w}} I \quad (9)$$

and

$$p_0 = -C\nabla_{\mathbf{x}} \cdot \mathbf{u}_0 - M\nabla_{\mathbf{x}} \cdot \bar{\mathbf{w}}, \quad (10)$$

where C and M are well defined scalar coefficients and J is a fourth rank tensor, all of which arise naturally within the two-scale analysis.

When equations (7)–(10) are compared with Biot’s equations, we find that the form of these equations is identical — once we have taken care to interpret each of these expressions in terms of the corresponding expressions in the other set of equations, as was done in the original publication by Burridge and Keller (1981). Thus, the two-space homogenization method produces exactly the same equations as Biot found using his variational approach. One advantage that the present method has is that it also produces definite formulas for the coefficients in these equations, so — at least in principle — model calculations can be done to produce a set of theoretical examples to study the quantitative behavior of these coefficients. As far as I am aware, this step has never been taken. It is not necessarily easy to compute these coefficients from the formulas, but it would nevertheless be an interesting exercise in the theory to do so.

In contrast, the volume averaging methods to be discussed next also produce the same equations, but they do not produce formulas for the coefficients. So the volume averaging approach is phenomenological, *i.e.*, producing a set of equations whose coefficients must be determined experimentally.

VOLUME AVERAGING METHODS

Pride et al. (1992) studied the way in which the equations of motion for sound traveling through a solid/fluid mixture can be derived from first principles when it is assumed that the solid is porous, but contains only a single type of mineral. The fluid is homogeneous and completely fills the pores. Various other authors have also studied volume averaging both for the simple single-constituent poroelasticity and for multi-constituent generalizations such as double-porosity poroelasticity (Tuncay and Corapcioglu 1995; Pride and Berryman 1998; Berryman and Pride 1998; Pride and Berryman 2003a; Pride and Berryman 2003b).

The averaging theorem

The averaging theorem used by all these authors is due to Slattery (1967) and is based on well-known mathematics (Green’s theorem and the divergence theorem) together with the idea that in relatively small regions *volume averages of spatial gradients* in statistically homogeneous media are presumably closely related to *gradients of volume averages*. But care must nevertheless be taken to account properly for behavior of the averaged quantities at points or surfaces where abrupt changes occur. In particular, when the quantity to be averaged exists on one side of an interface and does not exist on the other side, an interior interface term will contribute to the volume average of the derivative, but not to the derivative of the volume average.

Suppose that Q is a quantity to be averaged. Q can be a scalar, vector, or tensor. For convenience of the discussion, we will assume that the averaging volume is a finite sphere centered at position \mathbf{x} , although other choices are also possible (Pride and Berryman 1998). We label this volume $\Omega(\mathbf{x})$ and the surface of this volume is $\partial\Omega$. The exterior surface has two parts $\partial\Omega = \partial E_0 + \partial E_Q$, with ∂E_0 being the part where the quantity of interest Q vanishes identically and ∂E_Q being the part where $Q \neq 0$. For example, Q could represent some physical quantity in the pore space and 0 in the solid — or vice versa — depending on immediate interest. In addition to the exterior surface, there are also interior surfaces where Q changes abruptly to

zero and we label such surfaces ∂I_Q , for interior. The interior surface is the bounding surface for the region we will label Ω_Q , *i.e.*, the region wherein the quantity Q to be averaged is nonzero. With these definitions, Green's theorem gives

$$\int_{\Omega} \nabla Q d^3x = \int_{\Omega_Q} \nabla Q d^3x = \int_{\partial E_Q} \hat{\mathbf{n}}_Q Q dS + \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q dS, \quad (11)$$

where dS is the infinitesimal of the surface volume element, and $\hat{\mathbf{n}}_Q$ is the unit outward normal vector from the region containing nonzero Q . The main point of (11) is just that $\partial E_Q + \partial I_Q$ is the entire bounding surface of Q in the volume Ω . As an example of the meaning of this result, consider Q to be a vector quantity, take the trace of (11), and the result is just a statement of the well-known divergence theorem for vectors.

A second result of interest is that

$$\nabla \int_{\Omega} Q d^3x = \nabla \int_{\Omega_Q} Q d^3x = \int_{\partial E_Q} \hat{\mathbf{n}}_Q Q dS. \quad (12)$$

The result (12) follows from the fact that the volumes $\Omega(\mathbf{x})$ and $\Omega(\mathbf{x} + \delta\mathbf{x})$ contain virtually the same internal surfaces (in the limit $\delta\mathbf{x} \rightarrow 0$ they are obviously identical) and so these do not contribute to the gradient.

Combining these results finally gives

$$\int_{\partial E_Q} \hat{\mathbf{n}}_Q Q dS = \nabla \int_{\Omega} Q d^3x = \int_{\Omega} \nabla Q d^3x - \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q dS. \quad (13)$$

Dividing by the total volume $V = \int_{\Omega} d^3x$ (which is a constant scalar, since the size of Ω is the same everywhere) contained in Ω gives the averaging theorem:

$$\nabla \langle Q \rangle = \langle \nabla Q \rangle - \frac{1}{V} \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q dS. \quad (14)$$

Also note that the average $\langle Q \rangle$ is an average over the whole volume of Ω , while we also sometimes need to consider the partial average \bar{Q} , related to the full volume average by

$$\langle Q \rangle = \bar{v}_Q \bar{Q}, \quad (15)$$

where \bar{v}_Q is the volume fraction of Ω in which Q is nonzero.

Finally, although this dependence is often not explicitly shown or even mentioned, all the average quantities are in fact functions of the particular choice of averaging volume $\Omega(\mathbf{x})$. In principle, $\Omega(\mathbf{x})$ can be as large as the sample being studied, or as small as desired. The legitimacy of the averaging theorem itself does not depend at all on the size of this averaging volume. However, the usefulness of the resulting meso- or macro-scale equations does depend on this choice and so some intermediate size is generally picked for $\Omega(\mathbf{x})$. Too small of an averaging volume implies rapid fluctuations in the quantities of interest (like the fluid and solid dilatations), while a very large averaging volume implies all the coefficients in the equations are universal constants and, therefore, can prevent us from studying the effects of local inhomogeneities, whenever they are present.

Note, for example, that a most desirable (but not always correct) consequence of (14) is for the final surface integral to vanish identically. The vanishing of this integral is natural in statistically homogeneous media because the unit outward normal vector averages to zero if Q

is approximately constant on this surface. Vanishing of this surface integral is therefore often highly likely in reasonably homogeneous media in 3D (averaging over a 2D surface), still likely but somewhat less so in 2D (averaging over a 1D curve), and in general will not vanish in 1D (averaging over just two points) for any but some rather trivial model problems. So volume averaging methods should be replaced in 1D by exact methods such as, for example, Backus averaging (Backus 1962) for pertinent 1D applications.

In wave problems, when Ω is larger than the wavelength, the displacements will tend to average to zero, which is clearly an undesirable result. Pride *et al.* (1992) provide further discussion of criteria for choosing the size of the averaging volume. Thus, the choice of the averaging volume is often based on the same or similar issues normally used to pick an REV (representative elementary volume) in other methods, but we believe it is useful to maintain a strict distinction between these two concepts as the motivations for choices made are sometimes different.

Applications

Volume averaging has been applied successfully to derive the form of Biot's equations of poroelasticity (Pride *et al.* 1992), and more recently a wide variety of other up-scaling problems in double-porosity poroelasticity (Tuncay and Corapcioglu 1995; Pride and Berryman 1998; Berryman and Pride 1998; Pride and Berryman 2003a; Pride and Berryman 2003b). The method is well-suited to obtaining the forms of the equations, but needs to be supplemented when the values of the coefficients in the equations are required. The supplements can obviously be obtained experimentally, in which case the theory can be treated as a phenomenological one — like Biot's original formulation using Lagrangian variational principles. But, being phenomenological is not a serious limitation since most of the theories and equations of mathematical physics are in fact phenomenological in the same sense. There are some cases in poroelasticity where various other theoretical means, including some of those already mentioned here, such as effective medium theories and periodic cell homogenization theory, can be applied to obtain estimates of the constants (Mavko *et al.* 1998; Milton 2002). And in some special cases, exact results are known (Berryman and Milton 1991; Berryman and Pride 2002) for a two-component solid matrix. In these situations the problems can be solved explicitly and quite easily. In most other situations, it remains an open question whether the coefficients in the equations can be determined accurately either by exact or some well-controlled but approximate means.

CONCLUSIONS

Two methods of up-scaling coupled equations at the microscale to equations valid at the mesoscale and/or macroscale for fluid-saturated and partially saturated porous media have been discussed, compared, and contrasted. The two methods were: (1) two-scale and multiscale homogenization and (2) volume averaging. Both methods have advantages for some applications and disadvantages for others. Homogenization methods can give formulas for coefficients in the up-scaled equations, whereas volume averaging methods give the form of the up-scaled equations but generally must be supplemented with physical arguments and/or data in order to determine the coefficients. Homogenization theory requires a great deal of mathematical insight from the user in order to choose appropriate scalings for use in the resulting power-law expansions, while volume averaging requires more physical insight to motivate the steps needed to find coefficients. Homogenization often is performed on periodic models, while volume averaging does not require any assumption of periodicity and can therefore be related very directly to laboratory and/or field measurements. Validity of the homogenization process is

often limited to specific ranges of frequency – in order to justify the scaling hypotheses that are made – and therefore cannot be used easily over wide ranges of frequency. However, volume averaging methods can quite easily be used for wide band data analysis. So, we learn from these comparisons that a researcher in the theory of poroelasticity and its generalizations will benefit from being conversant with more than one of the methods to solve problems generally.

In this short review, we have not attempted to cover all methods that might be of interest and value for the applications considered. In particular, we have avoided discussion of ensemble averaging methods as well as other methods that might take the details of the spatial statistics of the complex heterogeneous media directly into account, or provide additional information about important corrections to the average equations. Recent publications by Drugan and Willis (1996) and Drugan (2003) suggest that such methods may also be of great value in the future.

ACKNOWLEDGMENTS

Work performed under the auspices of the U. S. Department of Energy by the University of California Lawrence Livermore National Laboratory under contract No. W-7405-ENG-48 and supported specifically by the Geosciences Research Program of the DOE Office of Energy Research within the Office of Basic Energy Sciences, Division of Chemical Sciences, Geosciences, and Biosciences.

REFERENCES

- Auriault, J.-L. (2002). “Upscaling heterogeneous media by asymptotic expansions.” *ASCE J. Engng. Mech.*, 128, 817–822.
- Auriault, J.-L. and Boutin, C. (1994). “Deformable porous media with double porosity III: Acoustics.” *Transport in Porous Media*, 14, 143–162.
- Auriault, J.-L. and Royer, P. (2002). “Seismic waves in fractured porous media.” *Geophysics*, 67, 259–263.
- Backus, G. E. (1962). “Long-wave elastic anisotropy produced by horizontal layering.” *J. Geophys. Res.*, 67, 4427–4440.
- Bensoussan, A., Lions, J.-L., and Papanicolaou, G. C. (1978). “Asymptotic analysis for periodic structures.” *Studies in Mathematics and Its Applications*, Vol. 5, Amsterdam. North-Holland.
- Berryman, J. G. and Milton, G. W. (1991). “Exact results for generalized Gassmann’s equations in composite porous media with two constituents.” *Geophysics*, 56, 1950–1960.
- Berryman, J. G. and Pride, S. R. (1998). “Volume averaging, effective stress rules, and inversion for microstructural response of multicomponent porous media.” *Int. J. Solids Struct.*, 35, 4811–4843.
- Berryman, J. G. and Pride, S. R. (2002). “Models for computing geomechanical constants of double-porosity materials from the constituents’s properties.” *J. Geophys. Res.*, 107 (B3), 10.1029/2000JB000108.
- Biot, M. A. (1941). “General theory of three-dimensional consolidation.” *J. Appl. Phys.*, 12, 155–164.
- Biot, M. A. (1956a). “Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range.” *J. Acoust. Soc. Am.*, 28, 168–178.
- Biot, M. A. (1956b). “Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher frequency range.” *J. Acoust. Soc. Am.*, 28, 179–191.
- Biot, M. A. (1962). “Mechanics of deformation and acoustic propagation in porous media.” *J. Appl. Phys.*, 33, 1482–1498.

- Brekhovskikh, L. M. (1980). *Waves in Layered Media*. Academic Press, New York.
- Budiansky, B. and O'Connell, R. J. (1976). "Elastic moduli of a cracked solid." *Int. J. Solids Struct.*, 12, 81–97.
- Burridge, R. and Keller, J. B. (1981). "Poroelasticity equations derived from microstructure." *J. Acoust. Soc. Am.*, 70, 1140–1146.
- Drugan, W. J. (2003). "Two exact micromechanics-based nonlocal constitutive equations for random linear elastic composite materials." *J. Mech. Phys. Solids*, 51, 1745–1772.
- Drugan, W. J. and Willis, J. R. (1996). "A micromechanics-based nonlocal constitutive equation and estimates of representative volume element size for elastic composites." *J. Mech. Phys. Solids*, 44, 497–524.
- Ewing, W. M., Jardetzky, W. S., and Press, F. (1957). *Elastic Waves in Layered Media*. McGraw-Hill, New York.
- Frenkel, J. (1944). "On the theory of seismic and seismoelectric phenomena in a moist soil." *J. Phys. (Moscow)*, 8, 230–241.
- Gassmann, F. (1951). "Über die elastizität poroser medien." *Veierteljahrsschrift der Naturforschenden Gesellschaft in Zurich*, 96, 1–23.
- Keller, J. B. (1977). "Effective behavior of heterogeneous media." *Statistical Mechanics and Statistical Method in Theory and Application*, U. Landman, ed., New York. Plenum, 631–644.
- Mavko, G., Mukerji, T., and Dvorkin, J. (1998). *The Rock Physics Handbook: Tools for Seismic Analysis in Porous Media*. Cambridge University Press, Cambridge.
- Milton, G. W. (2002). *The Theory of Composites*. Cambridge University Press, Cambridge.
- O'Connell, R. J. and Budiansky, B. (1977). "Viscoelastic properties of fluid-saturated cracked solids." *J. Geophys. Res.*, 82, 5719–5735.
- Pride, S. R. and Berryman, J. G. (1998). "Connecting theory to experiment in poroelasticity." *J. Mech. Phys. Solids*, 46, 719–747.
- Pride, S. R. and Berryman, J. G. (2003a). "Linear dynamics of double-porosity dual-permeability materials i. governing equations and acoustic attenuation." *Phys. Rev. E*, 68, 036603.
- Pride, S. R. and Berryman, J. G. (2003b). "Linear dynamics of double-porosity dual-permeability materials ii. fluid transport equations." *Phys. Rev. E*, 68, 036604.
- Pride, S. R., Gangi, A. F., and Morgan, F. D. (1992). "Deriving the equations of motion for porous isotropic media." *J. Acoust. Soc. Am.*, 92, 3278–3290.
- Sanchez-Palencia, E. (1980). "Non-homogeneous media and vibration theory." *Lecture Notes in Physics*, Vol. 127, New York. Springer-Verlag.
- Slattery, J. C. (1967). "Flow of visolastic fluids through porous media." *Am. Inst. Chem. Eng. J.*, 13, 1066–1071.
- Tuncay, K. and Corapcioglu, M. Y. (1995). "Effective stress principle for saturated fractured porous media." *Water Resources Res.*, 31, 3103–3106.
- Whitaker, S. (1999). *The Volume Averaging Method*. Kluwer Academic, Norwell, Mass.
- Whitaker, S. (2002). "Mechanics of composite solids." *ASCE J. Engng. Mech.*, 128, 823–828.
- Wood, B. D., Cherblanc, F., Quintar, M., and Whitaker, S. (2003). "Volume averaging for determining the effective dispersion tensor: Closure using periodic unit cells and comparison with ensemble averaging." *Water Resources Res.*, 39, 1210/10.1029/2002WR001723.