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Abstract: Higher-order basis functions have been constructed for surface-based differential forms that are used in engineering simulations. These surface-based forms have been designed to complement the volume-based forms present in EMSolve[1], a finite element code. The basis functions are constructed on a reference element and transformed, as necessary, for each element in space. Lagrange polynomials are used to create the basis functions. This approach is a necessary step in creating a hybrid finite-element/integral-equation time-domain code for electromagnetic analysis.

Keywords: Differential Forms, Integral Equations, Method of Moments, Time-domain, Frequency-domain.

1. Introduction

In this paper differential forms are used as a convenient way of classifying the various field types involved in solving Maxwell's equations. These differential forms have been used to create discrete differential forms volume basis functions, which are used in the unstructured finite element code EMSolve to solve time and frequency-domain engineering problems. As a step towards using differential forms to implement a hybrid finite-element/integral-equation formulation, differential forms surface basis functions are required.

Higher-order surface basis functions are implemented for each differential form. These basis functions exist on the reference element, with the calculus of differential forms providing the necessary transformation rules to map them to the real elements. Once the differential forms basis functions are constructed, it is relatively straightforward to implement new integral equations by examining the unknown quantities to determine which differential form type models them (see Table 1), implementing the appropriate Green's function, and constructing the appropriate element and global matrices. This procedure was followed for several different formulations to test the surface basis functions. The formulations used for validation include time and frequency-domain full-wave electromagnetics as well as an electrostatic formulation.

2. Bases Functions and Transformation Rules

We now present a construction of the surface differential forms and interpolatory basis functions to complement the volumetric forms previously defined in [1]. All surface elements in a physical

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Physical Properties	Units	Differential Form
Scalar Potential	V/m^{-0}	0-Form
Electric Field Intensity	V/m^{-1}	1-Form
Electric Surface Current	A/m^{-1}	1^-Form
Electric Surface Charge	C/m^{-2}	2-Form

Table 1. Physical quantities and their associated differential forms.

mesh are topologically equivalent to a reference element defined on the unit element in a reference coordinate system. We will explicitly define all mesh elements in this reference coordinate system as $\hat{\Omega} = \{\hat{x}_1, \hat{x}_2\}$. Now we let $\hat{\mathbf{r}} = \{\hat{x}_1, \hat{x}_2\}$ denote an arbitrary point in the reference coordinate system and $\mathbf{r} = \{x_1, x_2, x_3\}$ denote a point in physical space. There exists a mapping Φ from the reference element in $\hat{\Omega}$ to the physical element in Ω. This mapping and its Jacobian are defined as

$$\mathbf{r} = \Phi\left(\hat{\mathbf{r}}\right), \quad \mathrm{J}_{i,j} = \frac{\partial x_j}{\partial \hat{x}_i}. \tag{1}$$

Following the approach in [2] for finite element methods, all basis functions are defined on a reference element using the Lagrange polynomial of degree p, $L_i^p\left(x; X\right)$, where X is a set of (p+1) interpolation points, then transformed to the actual element. This approach is in contrast to that presented in [3], where the basis functions are formed on the actual element. The approach used here allows the reference space basis functions to be sampled and stored once per quadrature point used. The appropriate transformations are the only quantity that must be computed for every actual element.

Basis functions and their derivatives are formed for different differential forms. The 0-form basis functions (n) correspond to nodal basis functions at lowest order, and represent continuous scalar quantities across the surface mesh. The 1-form basis functions (w) correspond to curl-conforming finite element basis functions, suitable for representing vector quantities with continuous tangential components. Twisted 1-form (1^-forms) basis functions (Λ) are divergence-conforming edge basis functions, and can be used to represent vector quantities with continuous normal components. The 2-form basis functions (f) are scalar basis functions with no enforced continuity between elements. At lowest order, the 2-form basis functions are similar to scalar pulse basis functions. It should be noted that the derivative of an n-form differential form basis function is an (n+1)-form (e.g., the derivative of a 0-form basis function is a 1-form). All the basis functions are formed from Lagrange interpolating polynomials in the reference space (designated with a hat) and are transformed to the real elements. A summary of the transformations is given in Table 2.

Twisted 1-Forms

Twisted 1-forms were used to implement both frequency domain (FD) and time domain (TD) full wave electromagnetic integral equations. The FD and TD EFIE are written in the following compact form:

$$\hat{\mathbf{n}} \times \mathbf{E}^{inc}\left(\mathbf{r}, \begin{Bmatrix} \omega \\ t \end{Bmatrix}\right) = \hat{\mathbf{n}} \times \left(\begin{Bmatrix} j\omega \\ \dfrac{\partial}{\partial t} \end{Bmatrix} * \mathbf{A}(\mathbf{r}, \begin{Bmatrix} \omega \\ t \end{Bmatrix}) + \nabla \phi\left(\mathbf{r}, \begin{Bmatrix} \omega \\ t \end{Bmatrix}\right)\right) \tag{2}$$

Object	Transformation Rule	Units
0-form	$n \circ \Phi = \hat{n}$	m^{-0}
Grad 0-form	$(\nabla n) \circ \Phi = J^{-1}(\nabla \hat{n})$	m^{-1}
1-form	$(w) \circ \Phi = J^{-1}(\hat{w})$	m^{-1}
Curl 1-form	$(\nabla \times w) \circ \Phi = \dfrac{J^T}{\lvert J \rvert}(\nabla \times \hat{w})$	m^{-2}
1^-form	$(\Lambda) \circ \Phi = \dfrac{J^T}{\lvert J \rvert}(\hat{\Lambda})$	m^{-1}
Div 1^-form	$(\nabla \bullet \Lambda) \circ \Phi = \dfrac{1}{\lvert J \rvert}(\nabla \bullet \hat{\Lambda})$	m^{-2}
2-form	$(f) \circ \Phi = \dfrac{1}{\lvert J \rvert}(\hat{f})$	m^{-2}

Table 2. Differential form transformation rules.

The surface potentials are represented in terms of the twisted 1-forms integrated over the reference element as follows:

$$\begin{aligned}\mathbf{A}\left(\mathbf{r},\begin{Bmatrix}\omega\\ t\end{Bmatrix}\right) &= \int_S \begin{Bmatrix}\bar{\mathbf{G}}^{\mathbf{A}}(\mathbf{r}\,|\,\mathbf{r}',\omega)\\ \delta(\tau)/R\end{Bmatrix} \cdot \mathbf{J}\left(\mathbf{r}',\begin{Bmatrix}\omega\\ t\end{Bmatrix}\right) dS' \\ &= \sum_n J_n\left(\begin{Bmatrix}\omega\\ \tau\end{Bmatrix}\right) \int_{\hat{S}} \begin{Bmatrix}\bar{\mathbf{G}}^{\mathbf{A}}(\mathbf{r}\,|\,\mathbf{r}',\omega)\\ \delta(\tau)/R\end{Bmatrix} J^T \cdot \hat{\mathbf{\Lambda}}\, d\hat{S}\end{aligned} \tag{3}$$

and

$$\begin{aligned}\phi\left(\mathbf{r},\begin{Bmatrix}\omega\\ t\end{Bmatrix}\right) &= \int_S \begin{Bmatrix}G^{\phi}(\mathbf{r}\,|\,\mathbf{r}',\omega)\\ \int_0^{t-\lvert\mathbf{r}-\mathbf{r}'\rvert/c} \delta(\tau)/R\,d\tau\end{Bmatrix} \nabla \cdot \mathbf{J}\left(\mathbf{r}',\begin{Bmatrix}\omega\\ t\end{Bmatrix}\right) dS' \\ &= \sum_n J_n\left(\begin{Bmatrix}\omega\\ \tau\end{Bmatrix}\right) \int_{\hat{S}} \begin{Bmatrix}G^{\phi}(\mathbf{r}\,|\,\mathbf{r}',\omega)\\ \int_0^{t-\lvert\mathbf{r}-\mathbf{r}'\rvert/c} \delta(\tau)/R\,d\tau\end{Bmatrix} \nabla \cdot \hat{\mathbf{\Lambda}}\, d\hat{S}.\end{aligned} \tag{4}$$

Note that in (3) and (4) the determinant of the jacobian from the actual surface integration has cancelled with the determinant of the jacobian due to the transform of the basis from the reference element to the actual element. Applying a testing procedure with the twisted 1-forms results in the standard FD impedance matrix and the marching-on-in-time matrix systems,

$$\begin{Bmatrix}Z_{mn}(\omega)\\ Z_{mn}(0)\end{Bmatrix} \cdot J_n\left(\begin{Bmatrix}\omega\\ t\end{Bmatrix}\right) = V_m\begin{Bmatrix}\omega\\ t\end{Bmatrix} - \begin{Bmatrix}0\\ \sum_{j=1}^{T} Z(j)\end{Bmatrix} \cdot J_n\left(\begin{Bmatrix}\omega\\ \tau_j\end{Bmatrix}\right). \tag{5}$$

2-Forms

The 2-forms were used to represent surface charge on conducting bodies. The potential on a conductor can be written in terms of the surface charge as

$$\phi(r) = \frac{1}{4\pi\varepsilon_o}\int_S G^{\phi}(\mathbf{r}\,|\,\mathbf{r}',0)\,\rho_s(r')\,dS' = \sum_n \frac{\rho_n}{4\pi\varepsilon_o}\int_{\hat{S}} G^{\phi}(\mathbf{r}\,|\,\mathbf{r}',0)\,(\hat{f}\,)\,d\hat{S}\,, \tag{6}$$

where Galerkin testing is used to form a matrix system. The capacitance of the system can be calculated from stored energy which is given by

$$W_e = \frac{1}{2}\int_S \phi(\mathbf{r})\cdot\rho_s(\mathbf{r})\,dS = \sum_n \frac{\rho_n}{2}\int_{\hat{S}} \phi(\mathbf{r})(\hat{f}\,)d\hat{S}\,. \tag{7}$$

3. Results

The surface differential forms listed in Table 2 have been implemented within the EMSolve framework in a class library called BEMSTER. Results for both the 0-forms and 1-forms are not presented here, as they are more useful in hybrid finite-element/integral-equation problems and dielectric electrostatics formulations that are currently in progress. Results are presented for integral equations using the twisted 1-forms to model both frequency-domain and time-domain full-wave EM and using 2-forms to model charge as a primary quantity for the electrostatics integral formulations.

For the FD code, the magnitude of the centroid current on each element was compared with the centroid value from the frequency-domain code EIGER [4]. This current was compared for a unit PEC cube meshed with 150 quadrilaterals. The centroid values of the surface electric current show good agreement with the EIGER results as seen in Figure 1. For the time-domain case, the surface forms marching-on-in-time (MOT) algorithm achieved stable results for scattering from a unit PEC cube as illustrated in Figure 2. The figure contains the *x*-directed current at the center of the top of a unit cube when an *x*-directed gaussian plane wave is incident from above. These results are qualitatively similar to results from the MOT algorithm in [5].

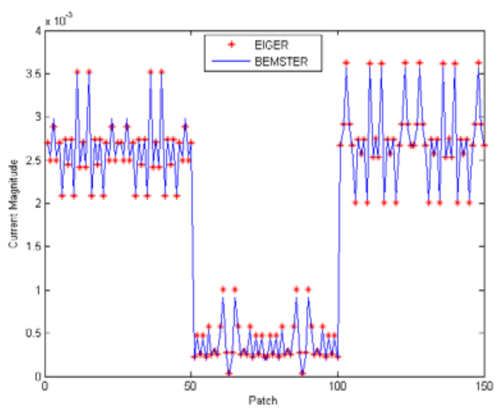

Figure 1. Frequency domain results comparison with EIGER.

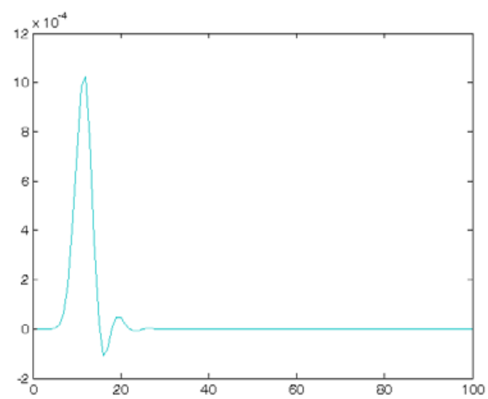

Figure 2. Stable time domain MOT algorithm based on surface differential forms.

The self-capacitance of both a unit sphere with 384 elements and a unit cube with 2400 elements were computed using the 2-forms. Table 3 contains the numerical results and the analytic solutions for each case.

Geometry	BEMSTER	Exact
Cube	73.36 pF.	73.03 pF.
Sphere	1.10e-10 F.	1.11e-10 F.

Table 3. Exact vs. numerical capacitance results based on 2-forms.

4. Summary

Higher-order basis functions have been constructed for surface-based differential forms that are used in engineering simulations. The approach taken allows the basis functions to be used in other engineering applications other than electromagnetics. These surface-based forms have been designed to complement the volume-based forms present in EMSolve, a finite-element code used at Lawrence Livermore National Laboratory. The basis functions are constructed on a reference element and transformed, as necessary, for each element in space. The results presented in this paper validate the surface basis function formulation. This approach is a necessary step in creating a hybrid finite-element/integral-equation time-domain code for electromagnetic analysis.

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