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The Radiation Transport Conundrum in Radiation Hydrodynamics

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The Radiation Transport Conundrum in Radiation Hydrodynamics

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Summary

- The conundrum in the title is whether to treat radiation in the lab frame or the comoving frame in a radiation-hydrodynamic problem

- Several of the difficulties are associated with combining a somewhat relativistic treatment of radiation with a non-relativistic treatment of hydrodynamics

- The principal problem is a tradeoff between easily obtaining the correct diffusion limit and describing free-streaming radiation with the correct wave speed

- The computational problems of the comoving-frame formulation in more than one dimension, and the difficulty of obtaining both exact conservation and full $\mathbf{u}/c$ accuracy argue against this method

- As the interest in multi-D increases, as well as the power of computers, the lab-frame method is becoming more attractive

- The Monte Carlo method combines the advantages of both lab-frame and comoving-frame approaches, its only disadvantage being cost

The goals of radiation hydrodynamics —

- We solve the nonrelativistic Euler equations, or their equivalent (ALE, etc.) for the material motion
- Radiation effects are included in the Euler equations as sources, balancing terms in the radiation momentum and energy equations
- Radiation transport should be calculated in a way that is accurate in both optically thick and optically thin regimes
- In the thick limit the radiation should tend to a Planck function when viewed in the comoving frame of the fluid, and the Euler and radiation equations should sum to the Euler equations with equilibrium radiation added to the material pressure and energy
- In the thin limit the radiation-matter coupling should become negligible and the radiation should obey the free-streaming transport equation in the laboratory frame

Why is this hard?

The problem arises because the radiation field is described by an intensity function $I(\mathbf{r}, \mathbf{n}, \nu, t)$, and there is a choice of frame implicit in this function, one of

- Laboratory frame — at rest with respect to the system as a whole
- Comoving frame — obtained from the former by a Lorentz transformation with the local fluid velocity

The direction vector $\mathbf{n}$ and the frequency ν change in the transformation, as does the value of I.

The transport operator is simple in the lab frame and the material-coupling terms are simple in the comoving frame, so both choices have good and bad aspects.

What the comoving frame is and is not

The comoving frame is the frame of reference for ν, $\mathbf{n}$ and I_ν, nothing more. It says nothing about Eulerian *vs.* Lagrangean hydrodynamics, *i.e.,* about the nature and motion of the computational mesh

At one time I advanced a 1-D approach using Riemannian-geometry methods with a comoving coordinate system. This turns out not to be necessary in 1-D and impossible in 2-D or 3-D, except for irrotational flows, a fatal limitation

The most successful comoving-frame approach is to proceed from the lab frame transport equation by applying Lorentz transformations

The Lab frame picture

The exact transport equation in the lab frame is

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n} \cdot \nabla I_\nu = j_\nu - k_\nu I_\nu$$

in which j_ν is the emissivity and k_ν is the absorptivity, and both include scattering.

The effects of fluid motion are buried in j_ν and k_ν.

Relativity or not?

The question arises whether or not to treat the kinematics relativistically in the radiation transport, when calculating the emission and absorption terms, or in the comoving-frame transport equation. The all-relativistic approach is consistent and recommended, but often we are called to couple radiation to non-relativistic hydrodynamics, which is the problem I want to consider. In this case there will be inconsistencies if $O(u^2/c^2)$ terms are retained in the kinematics.

The Doppler-aberration transformations

$$\nu = \nu_0 \gamma_u \left(1 + \frac{\mathbf{n}_0 \cdot \mathbf{u}}{c}\right)$$

$$\mathbf{n} = \frac{\gamma_u \mathbf{u}/c + \mathbf{n}_0 + (\gamma_u - 1)(\mathbf{n}_0 \cdot \mathbf{u})\mathbf{u}/u^2}{\gamma_u (1 + \mathbf{n}_0 \cdot \mathbf{u}/c)}$$

where "$_0$" quantities are in the comoving frame of the fluid, which moves with velocity $\mathbf{u}$, and $\gamma_u = (1 - u^2/c^2)^{-1/2}$. Make $u \ll c$ and get

$$\nu = \nu_0 \left(1 + \frac{\mathbf{n}_0 \cdot \mathbf{u}}{c}\right)$$

$$\mathbf{n} = \frac{\mathbf{n}_0 + \mathbf{u}/c}{1 + \mathbf{n}_0 \cdot \mathbf{u}/c}$$

Transformation of the intensity

The Lorentz transformation of the intensity is based on the principle that I_ν/ν^3 is a Lorentz invariant—

$$I_\nu = \left(\frac{\nu}{\nu_0}\right)^3 I_\nu^0$$

There are no simple rules for directly obtaining j_ν and k_ν in the lab frame. We must go to the comoving frame, obtain $j_{\nu_0}^0$ and $k_{\nu_0}^0$ using the constitutive relations for the material, possibly evaluating scattering terms expressed in terms of I_ν^0, then transform back to the lab frame to get j_ν and k_ν. No other approach, such as Taylor expansion, is suffi ciently accurate

The Euler equations with radiation coupling

$$g^0 = \int d\nu \int_{4\pi} d\Omega\, (j_\nu - k_\nu I_\nu)$$

$$\mathbf{g} = \frac{1}{c} \int d\nu \int_{4\pi} d\Omega\, \mathbf{n}(j_\nu - k_\nu I_\nu)$$

$$\frac{\partial}{\partial t}(\rho e + \frac{1}{2}\rho u^2) + \nabla\cdot(\rho \mathbf{u} h + \frac{1}{2}\rho \mathbf{u} u^2) = -g^0$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla\cdot(\rho \mathbf{u}\mathbf{u}) + \nabla p = -\mathbf{g}$$

Besides the usual symbols, h is the specifi c enthalpy of the material.

Important note: the radiation terms are all in the lab frame here!

Comoving-frame coupling terms

$$g_0^0 = \int d\nu_0 \int_{4\pi} d\Omega \, (j_\nu^0 - k_\nu^0 I_\nu^0)$$

$$\mathbf{g}_0 = \frac{1}{c} \int d\nu_0 \int_{4\pi} d\Omega \, \mathbf{n}_0 (j_\nu^0 - k_\nu^0 I_\nu^0)$$

from which it follows that

$$\begin{aligned} g^0 &= g_0^0 + \mathbf{u} \cdot \mathbf{g}_0 \\ \mathbf{g} &= \mathbf{g}_0 + \frac{\mathbf{u}}{c^2} g_0^0 \end{aligned}$$

to order u/c. The second term in the equation for $\mathbf{g}$ is problematic. It is the same order as the momentum addition to the material caused by the increase of the relative mass density when the material gains energy, *i.e.*, a purely relativistic effect

Radiation terms in the internal energy equation

If we neglect the $\mathbf{u}/c^2$ term in the $\mathbf{g}$ equation, then the total material energy and momentum equations combine to yield the internal energy equation

$$\frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho \mathbf{u} e) + p\nabla\cdot\mathbf{u} = -g^0 + \mathbf{u}\cdot\mathbf{g} \approx -g_0^0$$

Notice: the *internal* energy equation contains the radiation coupling in the comoving frame, while the *total* energy equation has the coupling term in the lab frame. We have to keep the frames straight!

Lab frame radiation moment equations

The energy and momentum conservation relations for radiation follow from the integrals of the transport equation over $\mathbf{n}$ and ν:

$$\begin{aligned}\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} &= \int d\nu \int_{4\pi} d\Omega \, (j_\nu - k_\nu I_\nu) = g^0 \\ \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} + c\nabla \cdot \mathsf{P} &= \frac{1}{c}\int d\nu \int_{4\pi} d\Omega \, \mathbf{n}(j_\nu - k_\nu I_\nu) = \mathbf{g}\end{aligned}$$

with the energy density, flux and pressure tensor defined by

$$E = \frac{1}{c}\int d\nu \int_{4\pi} I_\nu \, d\Omega \,, \quad \mathbf{F} = \int d\nu \int_{4\pi} \mathbf{n} I_\nu \, d\Omega \,, \quad \mathsf{P} = \frac{1}{c}\int d\nu \int_{4\pi} \mathbf{n}\mathbf{n} I_\nu d\Omega$$

Precise conservation is obtained by summing the moment equations with the Euler equations, but the radiation equations seem unlike the matter ones

Total conservation equations

$$\frac{\partial}{\partial t}(\rho e + E + \frac{1}{2}\rho u^2) + \nabla\cdot(\rho \mathbf{u} h + \mathbf{F} + \frac{1}{2}\rho \mathbf{u} u^2) = 0$$

$$\frac{\partial \rho \mathbf{u} + \mathbf{F}/c}{\partial t} + \nabla\cdot(\rho \mathbf{u}\mathbf{u} + \mathsf{P}) + \nabla p = 0$$

In order to achieve more symmetry between the matter and radiation terms it is necessary to cast the radiation moments in the comoving frame

Lorentz transformation of radiation moments

E, $\mathbf{F}$ and P are parts of a good stress-energy 4-tensor, and therefore the Lorentz transformation may be applied. It turns out to give

$$\begin{pmatrix} E & \mathbf{F}^T \\ \mathbf{F} & c^2\mathsf{P} \end{pmatrix} = \begin{pmatrix} E_0 + (2\mathbf{u}/c^2)\cdot\mathbf{F}_0 & (\mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0)^T \\ \mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0 & c^2\mathsf{P}_0 + \mathbf{F}_0\mathbf{u}^T + \mathbf{u}\mathbf{F}_0^T \end{pmatrix}$$

to order u/c. The corrections in E and P are often small; we will use the expression for $\mathbf{F}$ in terms of $\mathbf{F}_0$ in the total energy equation, which becomes

$$\frac{\partial}{\partial t}(\rho e + E + \frac{1}{2}\rho u^2) + \nabla\cdot\left(\rho\mathbf{u}h + \mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0 + \frac{1}{2}\rho\mathbf{u}u^2\right) = 0$$

The difference between $\mathbf{F}$ and $\mathbf{F}_0$ is just the convective radiation enthalpy flux needed to restore symmetry between matter and radiation

The comoving-frame approach

The comoving-frame method describes the radiation using $\mathbf{n}_0$ and ν_0, the direction vector and frequency as viewed by an observer comoving with the fluid. This is a particular case of using an arbitrary tetrad $\{e_a^\mu, a = 1, \ldots, 4\}$ as the basis for 4-momentum space at each point $\{x^\mu\}$ of spacetime, where the e_a^μ are any desired functions. Thus the 4-momentum components in the natural basis and in the tetrad basis are related by

$$p^\mu = e_a^\mu p^a$$

The functions e_a^μ form a 4×4 matrix of which the inverse is the matrix e_μ^a. The crucial objects related to the e_a^μ are the Ricci rotation coefficients Ω_{bc}^a defined in the following way: Let a vector with tetrad components M^a and natural components $M^\alpha = e_a^\alpha M^a$ be displaced parallel to itself along $dx^\alpha = e_a^\alpha dx^a$. Parallel displacement requires that $dM^\alpha = -\Gamma_{\beta\gamma}^\alpha M^\beta dx^\gamma$, in terms of the Christoffel coefficients Γ of the basic manifold. But the gradient in the tetrad functions also produces a change in the tetrad components for the displaced vector. The result is

$$dM^a = -\Omega_{bc}^a M^b dx^c \qquad \text{with} \qquad \Omega_{bc}^a = e_\alpha^a e_c^\gamma e_{b;\gamma}^\alpha = e_\alpha^a e_c^\gamma e_{b,\gamma}^\alpha + e_\alpha^a e_b^\beta e_c^\gamma \Gamma_{\beta\gamma}^\alpha$$

in which the comma and semicolon signify ordinary and covariant differentiation

The tetrad-component transport equation

We let $\mathscr{I} \propto I_\nu/\nu^3$, $\mathscr{a} \propto \nu k_\nu$ and $\mathscr{e} \propto j_\nu/\nu^2$ denote the invariant intensity, absorptivity and emissivity, respectively. Let s be an affine parameter on the photon's null geodesic, so $dx^\mu/ds = p^\mu$, where p^μ is the 4-momentum. Then the invariant transport equation is

$$\frac{d\mathscr{I}}{ds} = \mathscr{e} - \mathscr{a}\mathscr{I}$$

The derivative on the left is evaluated using the result just found for dp^a, with $p^\mu = e_a^\mu p^a$ —

$$e_a^\mu p^a \mathscr{I},\mu - \Omega^a_{bc} p^b p^c \frac{\partial \mathscr{I}}{\partial p^a} = \mathscr{e} - \mathscr{a}\mathscr{I}$$

The Ricci rotation coefficients derived from the Lorentz basis — 1

With the convention $x^0 = t$, $x^1 = x$, *etc.,* the choice for the tetrad given by the Lorentz-transformed natural basis is

$$(e_a^\mu) = \begin{pmatrix} \gamma_u & \gamma_u \mathbf{u}^\mathrm{T}/c^2 \\ \gamma_u \mathbf{u} & \mathsf{I} + (\gamma_u - 1)\mathbf{u}\mathbf{u}^\mathrm{T}/u^2 \end{pmatrix}$$

which in the low-velocity limit reduces to

$$(e_a^\mu) = \begin{pmatrix} 1 & \mathbf{u}^\mathrm{T}/c^2 \\ \mathbf{u} & \mathsf{I} \end{pmatrix}$$

and the inverse is approximately

$$(e_\mu^a) = \begin{pmatrix} 1 & -\mathbf{u}^\mathrm{T}/c^2 \\ -\mathbf{u} & \mathsf{I} \end{pmatrix}$$

The Ricci rotation coefficients derived from the Lorentz basis — 2

Then to $O(u)$

$$(e^{\mu}_{a,0}) = \begin{pmatrix} 0 & \mathbf{a}^{\mathrm{T}}/c^2 \\ \mathbf{a} & 0 \end{pmatrix}, \quad (e^{\mu}_{a,1}) = \begin{pmatrix} 0 & \partial_x \mathbf{u}^{\mathrm{T}}/c^2 \\ \partial_x \mathbf{u} & 0 \end{pmatrix}$$

$$(e^{\mu}_{a,2}) = \begin{pmatrix} 0 & \partial_y \mathbf{u}^{\mathrm{T}}/c^2 \\ \partial_y \mathbf{u} & 0 \end{pmatrix}, \quad (e^{\mu}_{a,3}) = \begin{pmatrix} 0 & \partial_z \mathbf{u}^{\mathrm{T}}/c^2 \\ \partial_z \mathbf{u} & 0 \end{pmatrix}$$

with $\mathbf{a} \equiv \partial \mathbf{u}/\partial t$ (part of the fluid acceleration), and since all these components are $O(u)$, the Ricci coefficients are seen to be

$$(\Omega^0_{bc}) = \begin{pmatrix} 0 & \mathbf{a}^{\mathrm{T}}/c^2 \\ 0 & (\nabla \mathbf{u})^{\mathrm{T}}/c^2 \end{pmatrix}, \quad (\Omega^1_{bc}) = \begin{pmatrix} a_x & 0 \\ \nabla u_x & 0 \end{pmatrix}$$

$$(\Omega^2_{bc}) = \begin{pmatrix} a_y & 0 \\ \nabla u_y & 0 \end{pmatrix}, \quad (\Omega^3_{bc}) = \begin{pmatrix} a_z & 0 \\ \nabla u_z & 0 \end{pmatrix}$$

Results from the Ricci coefficients

We see that unless $\mathbf{u}$ *is both uniform and constant in time the Ricci coefficients* Ω^a_{bc} *are not symmetric in the two lower indices* b *and* c*, which cannot happen for the natural basis in any coordinates, since the Christoffel coefficients are symmetric. In other words, in any other case the frame obtained by a pure boost cannot be a comoving coordinate frame*

The tetrad components of $p^b p^c \Omega^a_{bc}$ are found to be $\left(\mathbf{n}_0 \cdot \mathbf{a}/c + \mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot \mathbf{n}_0 \quad \mathbf{a} + c\mathbf{n}_0 \cdot \nabla\mathbf{u}\right)$, and this leads to the transport equation in a form similar to Buchler's (1983)

$$\left(1 + \frac{\mathbf{n}_0}{c} \cdot \mathbf{u}\right) \frac{1}{c}\frac{\partial I^0}{\partial t} + \left(\mathbf{n}_0 + \frac{\mathbf{u}}{c}\right) \cdot \nabla I^0$$
$$-\frac{\nu_0}{c}\left(\frac{\mathbf{a}}{c} + \mathbf{n}_0 \cdot \nabla\mathbf{u}\right) \cdot \nabla_{\nu_0 \mathbf{n}_0} I^0 + \frac{3}{c}\left(\frac{\mathbf{n}_0 \cdot \mathbf{a}}{c} + \mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot \mathbf{n}_0\right) I^0 =$$
$$j^0 - k^0 I^0$$

Critique of the comoving-frame equation

- Different suggestions have been made about the ordering of the terms in the CMF equation. Letting the characteristic length scale and time scale be L and T shows that the terms in the transport operator have orders of I^0/L, I^0/cT, $uI^0/(Lc)$ and $uI^0/(c^2T)$

- If T is $O(L/c)$ (radiation flow time scale) then the terms have order I^0/L and $uI^0/(Lc)$, and all terms are needed for first-order accuracy in u/c

- If T is $O(L/u)$ (fluid-flow time scale) then the orders are I^0/L, $uI^0/(Lc)$ and $u^2I^0/(c^2L)$, and the terms divided by c^2 are indeed second order in u/c and can be dropped for a first-order solution

- Both kinds of ordering have been, and are still, advocated by various authors

- The conservative solution is to include all the terms, or, better yet, use the fully relativistic form

Transforming the momentum-space gradient

Rather than considering I^0 as a function of the Cartesian tetrad components $\nu_0\mathbf{n}_0$, we can use spherical momentum coordinates: ν_0 and the angles implicit in $\mathbf{n}_0$. So instead of a momentum-space gradient term, we have a frequency-derivative term and an angle-derivative term:

$$\frac{\nu_0}{c}\left(\frac{\mathbf{a}}{c}+\mathbf{n}_0\cdot\nabla\mathbf{u}\right)\cdot\nabla_{\nu_0\mathbf{n}_0}I^0 \to$$

$$\frac{1}{c}\left(\frac{\mathbf{n}_0\cdot\mathbf{a}}{c}+\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0\right)\nu_0\frac{\partial I^0}{\partial\nu_0}+\frac{1}{c}\left(\frac{\mathbf{a}}{c}+\mathbf{n}_0\cdot\nabla\mathbf{u}\right)\cdot(\mathsf{I}-\mathbf{n}_0\mathbf{n}_0)\cdot\nabla_{\mathbf{n}_0}I^0$$

The former is the Doppler term and the latter is the aberration term. The factor $\mathsf{I}-\mathbf{n}_0\mathbf{n}_0$ is the perpendicular projector relative to $\mathbf{n}_0$ that converts $\nabla_{\mathbf{n}_0}$ into the gradient on the unit sphere

Frequency-integrated CMF moment equations

The transport equation including the $1/c^2$ terms leads to these equations for the frequency and angle moments:

$$\frac{\partial}{\partial t}\left(E_0 + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{F}_0\right) + \nabla\cdot(\mathbf{u}E_0 + \mathbf{F}_0) + \mathsf{P}_0:\nabla\mathbf{u} + \frac{\mathbf{a}}{c^2}\cdot\mathbf{F}_0$$
$$= \int d\nu\,(4\pi j_\nu^0 - k_\nu^0 c E_\nu^0) = g_0^0$$

and

$$\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{F}_0 + \mathbf{u}\cdot\mathsf{P}_0) + c\nabla\cdot\left(\mathsf{P}_0 + \frac{1}{c^2}\mathbf{u}\mathbf{F}_0\right) + \frac{\mathbf{a}}{c}E_0 + \frac{1}{c}\mathbf{F}_0\cdot\nabla\mathbf{u}$$
$$= -\int d\nu\,k_\nu^0\mathbf{F}_\nu^0 = c\mathbf{g}_0$$

Summing the CMF energy equation and the product of $\mathbf{u}$ with the momentum equation, and conversely, *then discarding the higher-order terms in u*, leads to

$$\frac{\partial}{\partial t}\left(E_0 + \frac{2}{c^2}\mathbf{u}\cdot\mathbf{F}_0\right) + \nabla\cdot\left(\mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0\right) = g_0^0 + \mathbf{u}\cdot\mathbf{g}_0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0\right) + \nabla\cdot\left[c\mathsf{P}_0 + \frac{1}{c}\left(\mathbf{u}\mathbf{F}_0 + \mathbf{F}_0\mathbf{u}\right)\right] = c\mathbf{g}_0 + \frac{1}{c}\mathbf{u}g_0^0$$

which are equivalent to the lab-frame moment equations given earlier

Global energy and momentum conservation are obeyed only to $O(u/c)$ when the comoving-frame equations of that order are used

Making conservative CMF moment equations

Dropping the "small" terms from the CMF moment equations leads to this set:

$$
\begin{aligned}
\frac{\partial E_0}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{u}E_0 + \mathbf{F}_0) + \mathsf{P}_0{:}\boldsymbol{\nabla}\mathbf{u} &= \int d\nu\,(4\pi j_\nu^0 - k_\nu^0 c E_\nu^0) = g_0^0 \\
c\boldsymbol{\nabla}\boldsymbol{\cdot}\mathsf{P}_0 &= -\int d\nu\, k_\nu^0 \mathbf{F}_\nu^0 = c\mathbf{g}_0
\end{aligned}
$$

which satisfy this energy conservation law

$$
\frac{\partial E_0}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}\,(\mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0) = g_0^0 + \mathbf{u}\cdot\mathbf{g}_0
$$

This system does exactly conserve energy and momentum, at the cost of not being hyperbolic—with unbounded propagation speed, but the correct diffusion limit. The other problem is that these moment equations do not follow accurately from any *form of the CMF transport equation. Castor, Buchler, Mihalas and Mihalas and others have advocated using this system*

The diffusion limit

For the limit $\lambda \to 0$, where λ is the radiation mean free path, there is an asymptotic relation for the comoving-frame intensity (the $\boxed{\text{boxed}}$ terms are omitted in the simplified equation):

$$I^0_{\nu_0} \sim B_{\nu_0} - \lambda_{\nu_0} \frac{dB_{\nu_0}}{dT} \left(\mathbf{n}_o \cdot \nabla T + \frac{1}{c}\frac{\partial T}{\partial t} + \frac{1}{c}\mathbf{u} \cdot \nabla T + \boxed{\frac{\mathbf{n}_0 \cdot \mathbf{u}}{c^2}\frac{\partial T}{\partial t} + \frac{\mathbf{n}_0 \cdot \mathbf{a}}{c^2} T} + \frac{\mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot \mathbf{n}_0}{c} T \right)$$

with the implications

- The energy density tends to thermal equilibrium in the comoving frame
- The flux is $O(\lambda)$ in the comoving frame
- Obtaining these results in a lab-frame calculation imposes constraints on the spatial differencing (see Mihalas and Auer [2001])

Solving CMF transport

A variety of methods have been used:

- With steady-state or backwards-time-differenced equations, the 1-D problem becomes a PDE in z (or r) and ν, and it can be solved as an initial-value problem in ν
- This requires a monotone velocity field, which is the case for many of the examples, but fails in general, or in 2-D or 3-D
- The gray (frequency-integrated) moment system removes the complexities of frequency- and angle-derivatives, but requires an auxiliary calculation to close the system of moments (*e.g.,* Eddington tensor or flux limiter)
- Either the transport equation or the angle-moment equations comprise a hyperbolic system (with the fluid equations) and the general Godunov procedure can be applied (*cf.,* Balsara 1998–1999, for the gray moment system)
- Godunov methods are based on exact conservation, and the simplified CMF equations are required to satisfy this

The mixed-frame expansion method

Introduced by Fraser (1966), and developed by Hsieh and Spiegel (1976), Mihalas and Klein (1982), and Lowrie, Morel and Hittinger (1999), this method uses the lab-frame equations with j_ν and k_ν replaced by $O(u/c)$ expansions involving j_ν^0 and k_ν^0:

$$
\begin{aligned}
j_\nu &\approx j_\nu^0 + \frac{1}{c}\mathbf{u}\cdot\mathbf{n}\left(2j_\nu^0 - \nu\frac{\partial j_\nu^0}{\partial \nu}\right) \\
k_\nu &\approx k_\nu^0 - \frac{1}{c}\mathbf{u}\cdot\mathbf{n}\left(k_\nu^0 + \nu\frac{\partial k_\nu^0}{\partial \nu}\right)
\end{aligned}
$$

apart from scattering; coherent isotropic scattering in the fluid frame leads to a messy expression for j_ν involving frequency derivatives of the intensity. All these expansions fail when u is comparable to or larger than a line width in velocity units—generally in any supersonic flow. For this reason this method is not used for problems involving lines

The Monte Carlo method

- The particles (bunches of photons) are tagged with the lab-frame frequency and direction
- Tracking is done in the lab frame, and the probability of an absorption or scattering event is computed in each zone by making a Doppler transformation of the particle when it enters a zone
- When an interaction event occurs, the particle is transformed into the fluid frame, so its energy and momentum may be deposited, or it may be scattered using the scattering matrix appropriate to the fluid frame; relativistic Compton scattering is easily accommodated using sampling methods
- If a scattering has occurred, the scattered photon is transformed back to the lab frame for further tracking

This method does the transport accurately to all orders of u/c

Implicit radiation-matter coupling

- In a majority of radhydro problems the coupling of matter and radiation is so strong that time-implicit methods must be used
- The velocity and density updates may be operator-split, but a simultaneous solution is needed for the radiation field and the material temperature
- A popular approach is to apply the Newton-Krylov technique to the radiation+temperature problem, with a pre-conditioner based on some simple radiation approximation
- This might be gray diffusion, or even a diagonal operator $\approx$ escape probability
- The accuracy ultimately depends on the “formal solution” that gives the radiation in terms of a known material field — the lab-frame method is a good candidate here
- For Monte Carlo, implicit methods like that of Fleck and Cummings are possibilities

Scorecard of the algorithms

Algorithm	Advantages	Disadvantages
pretend $\mathbf{u} = 0$	simplicity	radiation pressure and energy effects are lost entirely
mixed frame (lab-frame $\mathbf{u}$-expansion)	easy to solve the transport eq. (without scattering); exact conservation	fails for problems with lines; difficult to treat scattering; complexity; dense mesh in ν-$\mathbf{n}$
comoving frame moment eqs.	obtains diffusion limit; solve coupled radhydro problems with elliptic solvers; adapted to coupled RH Godunov method	frequency-dependent problem much more difficult to solve, esp. for non-monotone or multi-D flows; closures may be inaccurate; have to choose between conservation and full $\mathbf{u}/c$ accuracy
comoving-frame transport	obtains diffusion limit; no *ad hoc* closure	PDE difficult to solve for non-monotone or multi-D flows
Monte Carlo	exact apart from statistics	cost
lab-frame eqs. with exact (formal) sources	easy to solve; exact conservation	care is required with the sources and differencing to obtain diffusion limit; dense mesh in ν-$\mathbf{n}$

Summary

- The conundrum in the title is whether to treat radiation in the lab frame or the comoving frame in a radiation-hydrodynamic problem
- Several of the difficulties are associated with combining a somewhat relativistic treatment of radiation with a non-relativistic treatment of hydrodynamics
- The principal problem is a tradeoff between easily obtaining the correct diffusion limit and describing free-streaming radiation with the correct wave speed
- The computational problems of the comoving-frame formulation in more than one dimension, and the difficulty of obtaining both exact conservation and full $\mathbf{u}/c$ accuracy argue against this method
- As the interest in multi-D increases, as well as the power of computers, the lab-frame method is becoming more attractive
- The Monte Carlo method combines the advantages of both lab-frame and comoving-frame approaches, its only disadvantage being cost

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