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Isovector Pairing within the $so(5)$ Richardson-Gaudin Exactly Solvable Model

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Properties of a nucleon system interacting via isovector proton-neutron pairing can be described within the $so(5)$ generalized Richardson-Gaudin exactly-solvable model [1]. We present results for a system of 12 nucleon pairs within the full $fp + g_{9/2}$ shell-model space. We discuss coupling constant dependence of the pair energies, total energy of the system, and the occupation numbers.

1 Introduction

The pairing interaction has a central role in the description of the strongly correlated many-body systems. In the sixties R.W. Richardson showed that the eigenvalue problem of the pairing Hamiltonian can be formulated as a set of nonlinear algebraic equations [2, 3] . Using the properties of $su(2)$, which is the relevant algebra for the pairing Hamiltonian for one kind of fermions, he derived a set of equations coupling the pair energies, the single particle energies of the system, and the pairing strength. Making use of the Bethe ansatz method M. Gaudin [4] obtained a limit of these equations for large coupling constant. Years later Dukelsky,t Esebbag, and Schuck [5] proposed a generalization of the Richardson's equations. The resulting Richardson-Gaudin (RG) models are nowadays widely used for modelling strongly correlated systems in the condensed matter physics, nuclear physics, atomic and molecular physics. A review of the RG models and their applications is presented in ref. [6, 7].

In a recent paper Dukelsky et al. [1] presented a generalization of the RG models to those symmetry algebras that give rise to the well known Exactly Solvable Models (ESM) in nuclear physics. Special attention is paid to the *so(5)* model of isovector proton–neutron (pn) pairing in non–degenerate orbits. An exact solution of this problem was proposed first by Richardson [3], but later it was shown by Pan and Draayer [8], that it is incorrect for more then two pairs.

The aim of this paper is to apply the exact solution of the isovector pn-pairing model, proposed in [1] to describe a system of 12 pairs of fermions occupying the full $fp + g_{9/2}$ shell model space. In section II we will sketch the $so(5)$ isovector pn-pairing model. Later we will present the exact solutions of the RG equations for $T = 0$ and $T = 1$ isospin of the system as a functions of the coupling constant. At the end we will show the evolution of the total energy and the occupation numbers of the nucleons with the pairing strength.

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2 $so(5)$ Richardson-Gaudin model

The exactly solvable models have two main advantages: 1) They can describe in analytical or exactly solvable way a wide variety of phenomena; 2) They can be and have been used as testing ground for various many-body approaches.

We will consider a system of two kinds of fermions subject to pairing interaction. It could be a mixture of atoms, but we will think about protons and neutrons in a nucleus. Pairing is part of the nuclear Hamiltonian which takes into account the short-range nature of the effective nucleon-nucleon interaction in the nuclear medium. The main feature of the two-body pairing interaction is that it correlates nucleons that are in time reversal states into pairs.

The generalized $so(5)$ RG equations were derived by Dukelsky, Gueorguiev and Van Isacker [1] within the Gaudin–algebra approach [10]. Some of the $so(5)$ generators are given next. To obtain the full $so(5)$ algebre one has to add the relevant hermitian conjugate generators.
The $T = 1$ pair creation operators are:

$$b^{\dagger}_{-1,i} = n^{\dagger}_{i} n^{\dagger}_{\bar{i}}, \;\; b^{\dagger}_{0,i} = \frac{1}{\sqrt{2}} \left(n^{\dagger}_{i} p^{\dagger}_{\bar{i}} + p^{\dagger}_{i} n^{\dagger}_{\bar{i}}\right), \;\; b^{\dagger}_{1,i} = p^{\dagger}_{i} p^{\dagger}_{\bar{i}}, \tag{1}$$

where $b^{\dagger}_{-1,i}$, $b^{\dagger}_{0,i}$, and $b^{\dagger}_{1,i}$ create neutron-neutron, proton-neutron, and proton-proton pairs of nucleons occupying time reversal states.
Isospin-raising (T_+) and third component of the isospin (T_0) operators:

$$T_{+,i} = \frac{1}{\sqrt{2}} \left(p^{\dagger}_{i} n_i + p^{\dagger}_{\bar{i}} n_{\bar{i}}\right), \tag{2}$$

$$T_{0,i} = \frac{1}{2} \left(p^{\dagger}_{i} p_i + p^{\dagger}_{\bar{i}} p_{\bar{i}}\right) - \frac{1}{2} \left(n^{\dagger}_{i} n_i + n^{\dagger}_{\bar{i}} n_{\bar{i}}\right). \tag{3}$$

In the spherical shell model nucleons occupy single particle states with quantum numbers $\{j, m\}$. The index $\{i\}$ then corresponds to $\{j, m\}$ and $\bar{i}$ – to $\{j, \overline{m}\}$. Alternatively, due to the rotational symmetry, the angular moment $\{j\}$ can be used as a label instead of $\{i\}$, but then the corresponding degeneracy of $\Omega_i = (2j + 1)/2$ should be taken into account. For the $j = 1/2$ case ($\Omega_i = 1$) the relevant equations were derived by Links et al. [9].

Following the Ushveridize's procedure [10] and having in mind the properties of the $so(5)$ algebra one can derive the generalized RG equations for the $T = 1$ pn-pairing. They complete a set of $2N - T$ algebraic equations, where N is the number of pairs in the system, T is the isospin:

$$\begin{aligned} \frac{1}{g} &= \sum_{i=1}^{L} \frac{\Omega_i}{2\varepsilon_i - e_\alpha} + 2 \sum_{\beta(\neq\alpha)=1}^{N} \frac{1}{e_\alpha - e_\beta} - \sum_{\beta=1}^{N-T} \frac{1}{e_\alpha - w_\beta} \\ 0 &= \sum_{\beta(\neq\alpha)=1}^{N-T} \frac{1}{w_\alpha - w_\beta} - \sum_{\beta=1}^{N} \frac{1}{w_\alpha - e_\beta} \end{aligned} \tag{4}$$

Since $so(5)$ is a rank-two algebra, there are two sets of spectral parameters: e_α and w_β. Here ε_i is the single particle energy, Ω_i is the degeneracy of the single particle level and g is the strength of the pairing interaction. If a system of like particles is considered, one has to solve the original Richardson equations (for example, see [2, 3, 6]). The number of equations then equals the number of pairs in the system and there is just one type of spectral parameters (only e_α and no w_β terms).

Each solution of the generalized RG equations corresponds to an eigenstate of the pn-Hamiltonian. The spectral parameters e_α are interpreted as pair energies as in the case of the standard $su(2)$ pairing. The new set of parameters w_β are associated with the isospin $su_T(2)$ algebra. Their number is $N - T$, connected to the number of proton-neutron pairs and they don't appear in the expression for the eigenvalue of the pn-Hamiltonian.

In this presentation of the $so(5)$ pairing model we assume that all fermions are coupled to pairs, i.e. the seniority of the system is equal to zero. The model can easily incorporate broken pairs as well [5].

3 Properties of the generalized RG equations

We have studied the properties of the generalized RG equations in the case of 12 pairs of fermions occupying the full $fp + g_{9/2}$ shell. This is the way to describe the nucleus ^{64}Ge as a ^{40}Ca core and 12 valent nucleon pairs. The level scheme and the s.p. energies are introduced by Monnoye et al. [11]. The single particle energies are as follows: $\varepsilon_{f_{7/2}} = 0.0\ MeV$, $\varepsilon_{p_{3/2}} = 6.0\ MeV$, $\varepsilon_{f_{5/2}} = 6.25\ MeV$, $\varepsilon_{p_{1/2}} = 7.1\ MeV$ and $\varepsilon_{g_{9/2}} = 9.6\ MeV$. This is a shell-model problem which can not be solved exactly at present. If the strength of the pairing interaction is equal to zero the nucleons occupy the single particle orbits obeying the Pauli principle - there are 8 protons and 8 neutrons in the $f_{7/2}$ orbit and 4 protons and 4 neutrons in the $p_{3/2}$ sub-shell.

For solving the problem we consider a system of 12 complex algebraic equations for each pairing energy e_α and $12 - T$ for each w_β value. It can be proven that the pair energies and the w values can be either real or can appear in complex conjugated pairs. The system is solved numerically using the Newton-Raphson method. The first step of the numerical procedure is to determine the initial guess for the solution. We start solving the system for a very small value of the coupling constant. In the $T = 0$ case all pairing energies are complex conjugated. Their real parts are just below the s.p. levels which the pairs would occupy without pairing and the imaginary parts are evenly distributed around the zero within the interval of 0.2 MeV. The initial guess for the w values deviate slightly from the initial guess for the pair energies. Suppose we have found a solution for a very small value of the coupling constant, then we use it as an initial guess for the solution with a larger value of g and so on. In this way we study the evaluation of the solution of the RG equations as a function of the pairing strength.

Looking at the RG equations one can see that there are four types of singularities, which may appear when the denominator of a term goes to zero. The real problem is how to guess the initial value of the solution after a singular point. We use the derivatives of the solutions with g at the point where the solution still converges to predict its values for g beyond the singular point.

Figure 1 shows the evaluation of the solution of the RG equations for $T = 0$, which corresponds to the ground state of ^{64}Ge. The right panel shows the behavior of the real

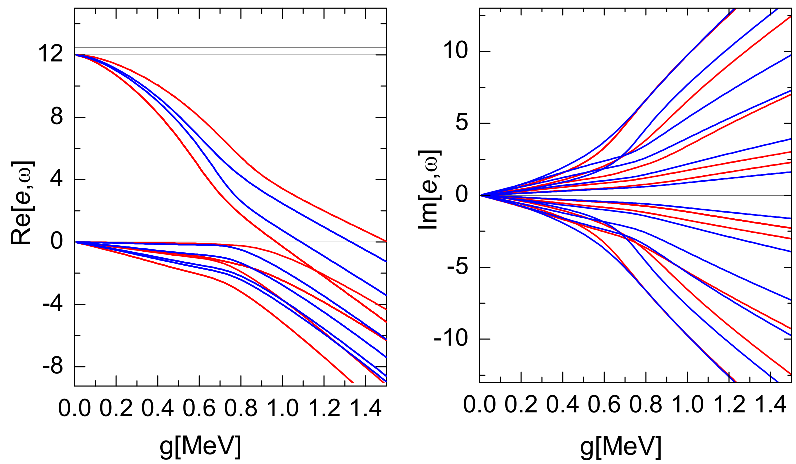

Figure 1: The spectral parameters e_α (red curves) and w_β (blue curves) for the $T = 0$ isospin. Black lines indicate the twice the s.p. energies ($g = 0$ pair energies).

part of the solutions and the left one – the evolution of the imaginary part. For $T = 0$ in the weak coupling regime 8 pairs of nucleons occupy the the $f_{7/2}$ orbit and 4 - the $p_{3/2}$. The pair energies appear in complex conjugated pairs thus there are two red line starting from the $p_{3/2}$ level and 4 red lines emerging from the $f_{7/2}$. The same holds for the w values as well. When the pairing strength increases, first the levels close to the Fermi surface are influenced. The pairing interaction is attractive and the real parts of all pair energies decrease, when the coupling strength increases. The imaginary parts get spread to larger absolute values. In this case there are no singularities observed.

The $T = 1$ case corresponds to an excited state of ^{64}Ge (see figure 2). There are 12 pair energies and 11 w values. In the weak coupling regime 8 pairs are close to the $f_{7/2}$ level, 3 - to the $p_{3/2}$ level and one - to the $f_{5/2}$ level. The pair energy for the pair close to the $f_{5/2}$ level is real and doesn't change significantly when the coupling increases. There is no w spectral parameter close to this level. There is one real and two complex conjugated pair energies just below the $p_{3/2}$ level. The same holds for the three w values close to this level. The solutions attached to the $f_{7/2}$ level appear in complex conjugated pairs. When the pairing constant g increases one reaches the point when a real w value become close to the real part of two complex conjugated w value and at the same point their imaginary parts vanish. A singularity occurs and the system for $g = 0.795\ MeV$ doesn't have a stable solution. One has to propose again an initial guess for the solutions for values of g beyond the critical point using the derivatives of the last stable set of solutions. Test for the quality of the initial guess is not just the existence of a solution. One has to be sure that the obtained solution describes the same state of the system. And there are two criteria for that which will be discussed in

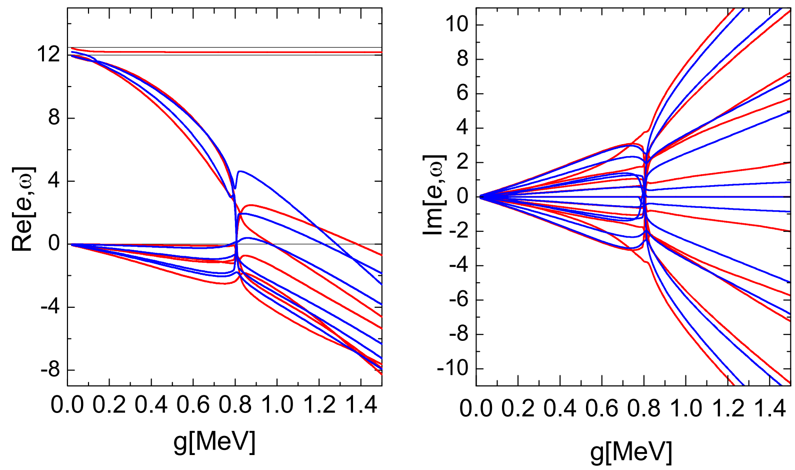

Figure 2: The spectral parameters e_α (red curves) and w_β (blue curves) for the $T = 1$ isospin. Black lines indicate twice the s.p. energies ($g = 0$ pair energies).

section 4.

4 Total energy and occupation numbers

In this section we will discuss the results for the total energy and the occupation numbers of the system.

The basic observable for each nuclear system is the total energy. It is calculated as the expectation value of the Hamiltonian and in the RG model it is simply the sum of the pair energies:

$$E = \sum_{i=1}^{N} e_i \tag{5}$$

Figure 3 illustrates the behavior of the total energy as a function of the pairing strength. In the weak coupling regime ($g = 0.1 MeV$) the energy of the system is $E = 47.81\ MeV$ for the ground $T = 0$ state and $E = 48.30\ MeV$ for the excited $T = 1$ state. As the attraction of the nucleons in time reversed states increases, the total energy of the system goes from positive to negative.

As it has been mentioned in section 3, one has to confirm that the solution of the RG equations after a critical point ($g > g_c$) still describes the same state of the system as before the critical point ($g < g_c$). One such criterium is the smoothness of the total energy of the system as a function of g. Another criterium is the smooth behavior of the occupation numbers when the value of g passes through the critical value (g_c).

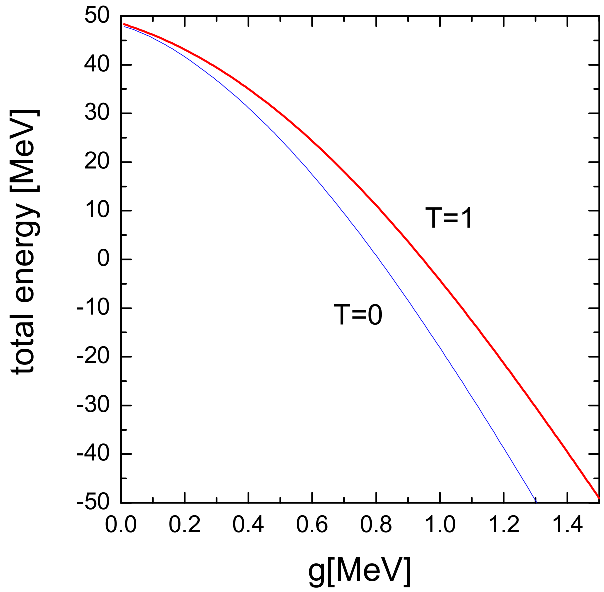

Figure 3: Total energy of the system for the ground $T=0$ state (blue curve) and for the excited $T=1$ state (red curve).

The occupation probabilities are important observables in the exactly solvable models. They are defined as expectation values of the number operators and can be obtained from the integrals of motion using the Hellmann-Feynmann theorem. The occupation numbers depend on the derivatives of the pair energies with respect to the coupling constant:

$$\eta_i = g^2 \sum_{\alpha=1}^{N} \frac{\Omega_i}{(2\varepsilon_i - e_\alpha)^2} \frac{\partial e_\alpha}{\partial g}, \tag{6}$$

One can receive derivatives of the pair energies solving a system of linear equations obtained by differentiation of the RG equations with respect to g:

$$\left[\sum_{i=1}^{L} \frac{\Omega_i}{(2\varepsilon_i - e_\alpha)^2} - \sum_{\beta=1}^{N-T} \frac{1}{(e_\alpha - w_\beta)^2} + 2 \sum_{\beta(\neq\alpha)=1}^{N} \frac{1}{(e_\alpha - e_\beta)^2} \right] \frac{\partial e_\alpha}{\partial g} -$$

$$2\sum_{\beta(\neq\alpha)=1}^{N}\frac{1}{(e_\alpha-e_\beta)^2}\frac{\partial e_\beta}{\partial g}+\sum_{\beta=1}^{N-T}\frac{1}{(e\_\alpha-w_\beta)^2}\frac{\partial w_\beta}{\partial g}=\frac{1}{g^2}$$

$$\left[\sum_{\beta=1}^{N}\frac{1}{(w_\alpha-e_\beta)^2}-\sum_{\beta(\neq\alpha)=1}^{N-T}\frac{1}{(w_\alpha-w_\beta)^2}\right]\frac{\partial w_\alpha}{\partial g}+ \tag{7}$$

$$\sum_{\beta(\neq\alpha)=1}^{N-T}\frac{1}{(w_\alpha-w_\beta)^2}\frac{\partial w_\beta}{\partial g}-\sum_{\beta=1}^{N}\frac{1}{(w_\alpha-e_\beta)^2}\frac{\partial e_\beta}{\partial g}=0$$

In figure 4 the evolution of the occupation numbers as functions of the pairing strength is displayed. At weak coupling the occupation numbers of the hole s.p. states for the ground state of the system are almost equal but less then the corresponding degeneracies. The particle levels are empty. When the coupling constant increases the pairs start to populate the particle s.p. levels as well and the depletion of the Fermi sea increases. This configuration is not allowed for the $T = 1$ case due to the Pauli principle.

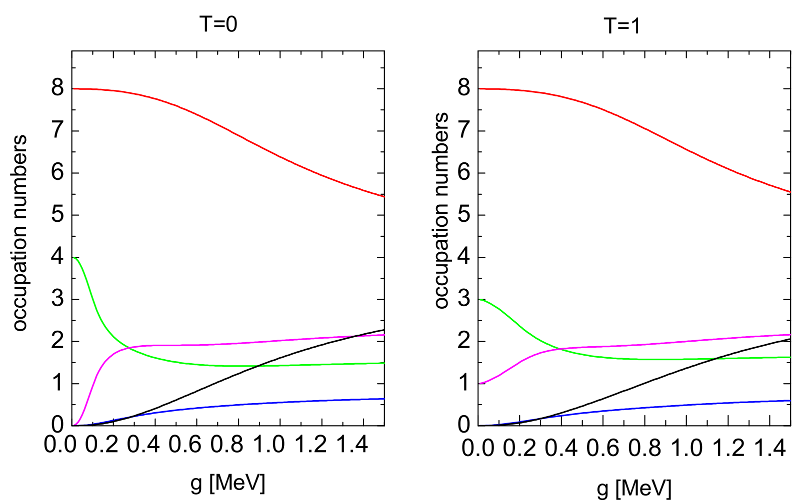

Figure 4: Occupation numbers of the s.p. orbits: $f_{7/2}$ (red curve), $p_{3/2}$(green curve), $f_{5/2}$ (magenta curve), $p_{1/2}$ (blue curve) and $g_{9/2}$ (black curve) for the ground $T = 0$ state (left panel) and for the excited $T = 1$ state (right panel).

In the excited $T = 1$ state for small values of g one fermion occupies the particle $f_{5/2}$ orbit and three are left in the $p_{3/2}$ one. When the coupling increases the occupation numbers of the particle levels also increase. It is interesting to notice that for strong coupling ($g \approx 0.3$ MeV for $T = 0$ and $g \approx 0.4$ MeV for $T = 1$) the $f_{5/2}$ orbit becomes more populated then the $p_{3/2}$ one.

5 Summary

In summary, we have investigated the solutions of the generalized $so(5)$ RG equations. As an example we have discussed the properties of a system of 12 fermions within the full $fp + g_{9/2}$ configuration space, interacting via isovector proton-neutron pairing. We belief that the numerical procedure which we have used can be applied to tread very large systems, which is of great importance in the condensed–matter physics in addressing phenomena like high T_c–superconductivity [12, 13].

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