LAWRENCE
LIVERMORE
NATIONAL
LABORATORY

UCRL-PROC-228221

Parallel auxiliary space AMG for definite Maxwell problems

Tz. V. Kolev, P. S. Vassilevski

February 21, 2007

Oberwolfach Institute workshop on "Computational Electromagnetism and Acoustics"
Oberwolfach, Germany
February 3, 2007 through February 10, 2007

This work was performed under the auspices of the U. S. Department of Energy by University of California, Lawrence Livermore National Laboratory under Contract W-7405-Eng-48.

Parallel auxiliary space AMG for definite Maxwell problems

Tzanio V. Kolev

(joint work with Panayot S. Vassilevski)

Motivated by the needs of large mullti-physics simulation codes, we are interested in algebraic solvers for the linear systems arising in time-domain electromagnetic simulations. Our focus is on finite element discretizations, and we are developing scalable parallel preconditioners which employ only fine-grid information, similar to algebraic multigrid (AMG) for diffusion problems. In the last few years, the search for efficient algebraic preconditioners for $\boldsymbol{H}(\mathrm{curl})$ bilinear forms has intensified. The attempts to directly construct AMG methods had some success, see [12, 1, 7]. Exploiting available multilevel methods on auxiliary mesh for the same bilinear form led to efficient auxiliary mesh preconditioners to unstructured problems as shown in [4, 8]. A computationally more attractive approach was recently proposed by Hiptmair and Xu [5]. In contrast to the auxiliary mesh idea, the method in [5] uses a nodal $\boldsymbol{H}^1$-conforming auxiliary space on the same mesh. This significantly simplifies the computation of the corresponding interpolation operator.

In the present talk, we consider several options for constructing unstructured mesh AMG preconditioners for $\boldsymbol{H}(\mathrm{curl})$ problems and report a summary of computational results from [10, 9]. Our approach is slightly different than the one from [5], since we apply AMG directly to variationally constructed coarse-grid operators, and therefore no additional Poisson matrices are needed on input. We also consider variable coefficient problems, including some that lead to a singular matrix. Both type of problems are of great practical importance and are not covered by the theory of [5].

We are interested in solving the following variational problem stemming from the definite Maxwell equations:

$$(1) \qquad \text{Find } \mathbf{u} \in \boldsymbol{V}_h \; : \quad (\alpha \, \mathrm{curl}\, \mathbf{u}, \mathrm{curl}\, \mathbf{v}) + (\beta\, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})\,, \quad \text{for all } \mathbf{v} \in \boldsymbol{V}_h\,.$$

Here we consider $\alpha > 0$ and $\beta \geq 0$ which are scalar coefficients, but extensions to (semi)definite tensors are possible. We allow β to be zero in part or the whole domain (in which case the resulting matrix is only semidefinite and for solvability, the right-hand side should be chosen to satisfy compatibility conditions). Let $\boldsymbol{A}_h$ be the stiffness matrix corresponding to (1), where $\boldsymbol{V}_h$ is the (lowest order) Nédélec space associated with a triangulation $\mathcal{T}_h$.

Let S_h be the space of continuous piecewise linear finite elements associated with the same mesh $\mathcal{T}_h$ as $\boldsymbol{V}_h$, and $\boldsymbol{S}_h$ be its vector counterpart. Let G_h and $\boldsymbol{\Pi}_h$ be the matrix representations of the mapping $\varphi \in S_h \mapsto \nabla\varphi \in \boldsymbol{V}_h$ and the nodal interpolation from $\boldsymbol{S}_h$ to $\boldsymbol{V}_h$, respectively. Note that G_h has as many rows as the number of edges in the mesh, with each row having two nonzero entries: $+1$ and -1 in the columns corresponding to the edge vertices. The sign depends on the orientation of the edge. Furthermore, $\boldsymbol{\Pi}_h$ can be computed based only on G_h and on the coordinates of the vertices of the mesh.

The auxiliary space AMG preconditioner for $\boldsymbol{A}_h$ is a subspace correction method utilizing the subspaces $\boldsymbol{V}_h$, $G_h S_h$, and $\boldsymbol{\Pi}_h \boldsymbol{S}_h$. Its additive form reads (cf. [13])

$$\Lambda_h^{-1} + G_h B_h^{-1} G_h^T + \boldsymbol{\Pi}_h \boldsymbol{B}_h^{-1} \boldsymbol{\Pi}_h^T \,, \tag{2}$$

where Λ_h is a smoother for $\boldsymbol{A}_h$, while B_h and $\boldsymbol{B}_h$ are efficient preconditioners for $G_h^T \boldsymbol{A}_h G_h$ and $\boldsymbol{\Pi}_h^T \boldsymbol{A}_h \boldsymbol{\Pi}_h$ respectively. Since these matrices come from elliptic forms, the preconditioner of choice is AMG (especially for unstructured meshes).

If β is identically zero, one can skip the subspace correction associated with G_h, in which case we get a two-level method.

The motivation for (2) is that any finite element function $\mathbf{u}_h \in \boldsymbol{V}_h$ allows for decomposition of the form (cf., [5]) $\mathbf{u}_h = \mathbf{v}_h + \boldsymbol{\Pi}_h \mathbf{z}_h + \nabla \varphi_h$ with $\mathbf{v}_h \in \boldsymbol{V}_h$, $\mathbf{z}_h \in \boldsymbol{S}_h$ and $\varphi_h \in S_h$ such that the following stability estimates hold,

$$h^{-1} \|\mathbf{v}_h\|_0 + \|\mathbf{z}_h\|_1 \leq C \, \| \operatorname{curl} \mathbf{u}_h \|_0 \quad \text{and} \quad \|\nabla \varphi_h\|_0 \leq C \, \|\mathbf{u}_h\|_0. \tag{3}$$

A parallel solution algorithm based on (2) was implemented in the *hypre* library [6], under the name AMS (Auxiliary space Maxwell Solver). The internal AMG V-cycles employ *hypre*'s algebraic multigrid solver BoomerAMG [2]. Our results on unstructured meshes in two and three dimensions, including problems with variable coefficients and zero conductivity, clearly demonstrate the scalability of this preconditioner on hundreds of processors. A sample of the numerical experiments to be presented is shown in Table 1, using the following notation: np is the number of processors in the run, N is the global size of the problem, and t denotes the average time to solution (in seconds) on a machine with 2.4GHz Xeon processors.

References

[1] P. Bochev, C. Garasi, J. Hu, A. Robinson and R. Tuminaro, *An improved algebraic multigrid method for solving Maxwell's equations*, SIAM J. Sci. Comp. **25** (2003), 623–642.

[2] V. Henson and U. Yang, *BoomerAMG: A parallel algebraic multigrid solver and preconditioner*, Appl. Numer. Math. **41** (2002), 155–177.

[3] R. Hiptmair, *Multigrid method for Maxwell's equations*, SIAM J. Numer. Anal. **36** (1999), 204–225.

[4] R. Hiptmair, G. Widmer, and J. Zou, *Auxiliary space preconditioning in H(curl)*, Numerische Mathematik **103** (2006), 435–459.

[5] R. Hiptmair and J. Xu, *Nodal auxiliary space preconditioning in H(curl) and H(div) spaces*, Research Report No. **2006-09** (May 2006). Seminar für Angewandte Mathematik, Eidgenössische Technische Hochschule, CH-8092 Zürich, Switzerland.

[6] *hypre*: High performance preconditioners, `http://www.llnl.gov/CASC/hypre/`.

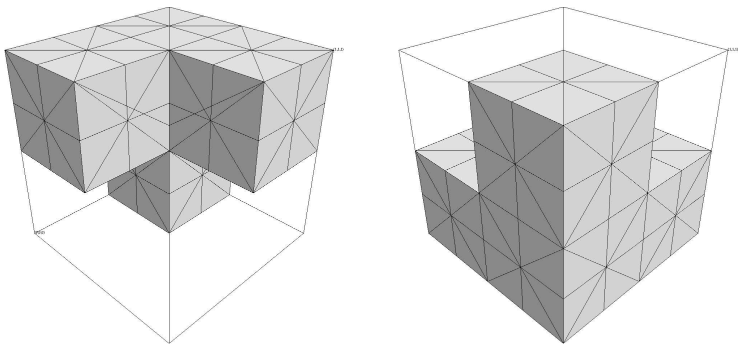
[7] J. Jones and B. Lee, *A multigrid method for variable coefficient Maxwell's equations*, SIAM J. Sci. Comp. **27** (2006), 1689–1708.

[8] Tz. Kolev, J. Pasciak and P. Vassilevski, *H(curl) auxiliary mesh preconditioning*, Num. Lin. Alg. Appl. (2007), to appear.

[9] Tz. Kolev and P. Vassilevski, *Some experience with a H1-based auxiliary space AMG for H(curl) problems.* LLNL Technical Report **221841** (June, 2006).

[10] Tz. Kolev and P. Vassilevski, *Parallel H1-based auxiliary space AMG solver for H(curl) problems.* LLNL Technical Report **222763** (July, 2006).

[11] J. Pasciak and J. Zhao, *Overlapping Schwarz methods in H(curl) on polyhedral domain*, East West J. Numer. Anal. **10** (2002), 221–234.



np	N	p									t
		-8	-4	-2	-1	0	1	2	4	8	
1	83,278	9	9	9	9	9	9	10	11	11	5s
2	161,056	10	10	10	10	10	10	10	11	11	9s
4	296,032	11	12	12	12	11	11	12	13	13	9s
8	622,030	13	13	13	12	12	12	13	15	14	12s
16	1,249,272	13	13	13	13	13	13	13	15	14	13s
32	2,330,816	15	15	15	15	15	15	15	16	15	14s
64	4,810,140	16	16	16	16	16	15	16	18	17	17s
128	9,710,856	16	16	16	16	16	16	16	17	17	23s
256	18,497,920	19	19	19	19	19	19	19	21	20	27s
512	37,864,880	21	20	20	20	20	20	20	23	22	32s
1024	76,343,920	20	20	20	20	20	20	20	21	21	56s

TABLE 1. Number of AMS-PCG iterations with tolerance 10^{-6}, for the problem (1) on the unit cube with $\alpha = 1$, and $\beta \in \{1, 10^p\}$ having different values in the shown regions (cf. [3]).

[12] S. Reitzinger and J. Schöberl, *An algebraic multigrid method for finite element discretization with edge elements*, Num. Lin. Alg. Appl. **9** (2002), 215–235.

[13] J. Xu, *The auxiliary space method and optimal preconditioning techniques for unstructured grids*, Computing **56** (1996), 215–235.