



LAWRENCE
LIVERMORE
NATIONAL
LABORATORY

On Ideal Stability of Cylindrical Localized Interchange Modes

M. V. Umansky

May 21, 2007

Contributions to Plasma Physics

Disclaimer

This document was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor the University of California nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or the University of California, and shall not be used for advertising or product endorsement purposes.

On ideal stability of cylindrical localized interchange modes

M.V.Umansky

Lawrence Livermore National Laboratory, Livermore, CA 94551

15 May 2007

Abstract

Stability of cylindrical localized ideal pressure-driven interchange plasma modes is revisited. Converting the underlying eigenvalue problem into the form of the Schrödinger equation gives a new simple way of deriving the Suydam stability criterion and calculating the growth rates of unstable modes. Near the marginal stability limit the growth rate is exponentially small and the mode has a double-peak structure.

Stability of ideal cylindrical interchange pressure-driven plasma modes is a classical problem. The well-known Suydam stability criterion [1–3] written as

$$D_s \equiv -\frac{q^2}{q'^2} \frac{\beta'}{r} < 1/4, \quad (1)$$

states the necessary conditions for stability, thus for $D_s \geq 1/4$ unstable modes must exist. Here the notation is $\beta = 8\pi p/B_z^2$, $q = rB_z/R_0B_\theta$, where R_0 is the radius of equivalent torus, prime stands for d/dr .

Previous analyses indicated exponentially small growth rates as the marginal stability boundary is approached [4–7].

In this paper we present analysis of ideal cylindrical interchange modes looking at the problem at a new angle. By choosing an appropriate transformation of independent and dependent variables we transform the underlying eigenvalue problem to the Schrödinger equation form. This allows to analyze the problem and calculate the growth rates of unstable modes using simple calculations based on elementary quantum mechanics.

We start with the equation analyzed in [4]

$$\frac{d}{dx} \left[(\hat{\omega}^2 - x^2) \frac{d\phi}{dx} \right] - [\hat{\omega}^2 - x^2 + D_s] \phi = 0 \quad (2)$$

Here $\hat{\omega} = \omega/(V_A/L_s)$ is normalized frequency and $x = k_\perp(r - r_*)$ is normalized radial coordinate measured from the resonant surface. As was pointed out in [4] this equation can be derived from ideal MHD screw pinch model for low β or incompressible plasma. Essentially the same equation is derived in [3] from high-beta reduced MHD model in cylindrical geometry.

As our interest is the unstable situation we rewrite it using $\hat{\gamma}^2 = -\hat{\omega}^2$, where $\hat{\gamma}^2$ is assumed positive

$$-\frac{d}{dx} \left[(x^2 + \hat{\gamma}^2) \frac{d\phi}{dx} \right] + [x^2 + \hat{\gamma}^2]\phi = D_s \phi \quad (3)$$

Eq. (3) is in the form of the classical Sturm-Liouville problem that can be transformed by Liouville's transformation into the Liouville normal form (or Schrödinger form) [8].

For our case the transformation is as follows

$$\begin{aligned} x &= \hat{\gamma} \sinh t \\ v(t) &= \phi(x(t)) \sqrt{\hat{\gamma} \cosh t} \end{aligned} \quad (4)$$

Then our eigenvalue problem, Eq.(3), transformed in the normal form becomes

$$-\frac{d^2v}{dt^2} + U(t)v = Ev, \quad (5)$$

where $E = D_s - 1/4$ and the potential energy function is

$$U(t) = \hat{\gamma}^2 \cosh^2 t + \frac{1}{4 \cosh^2 t}, \quad (6)$$

see Fig. (1).

An important issue is the asymptotic behavior of $v(t)$ at infinity. The original equation, Eq. (3), in the limit $x \rightarrow \infty$ becomes

$$\frac{1}{x^2} \frac{d}{dx} \left[x^2 \frac{d\phi}{dx} \right] = \phi \quad (7)$$

This equation has solutions $\phi(x) \rightarrow \exp(\pm x)/x$, so it decays at infinity faster than $1/x$, which in the t coordinate corresponds to decaying at infinity faster than $\exp(\pm t)$. Asymptotically, for $t \rightarrow \infty$, the transformed function is $v(t) = \phi(x(t)) \sqrt{\hat{\gamma} \cosh t} \rightarrow \phi(t) \exp(t/2)$, so we conclude that it decays to zero at infinity, as is normally assumed for solutions of the Schrödinger equation.

Now, as the potential energy function is positive, $U(t) > 0$, then using the conventional argument of quantum mechanics we infer that the energy E must be positive [9]. Thus one can immediately arrive at the conclusion that for an unstable solution ($\hat{\gamma}^2 > 0$ was our initial assumption) it is necessary that $E = D_s - 1/4 > 0$. This means that $E = D_s - 1/4 \leq 0$ is sufficient for stability of considered localized modes.

On the other hand, using physical arguments one can prove the Suydam necessary condition, Eq. (1), in a new way. One just needs to observe that for any $E > 0$ some $\hat{\gamma}^2 > 0$ can be found such that E is the ground state energy for Eq. (5). Below we demonstrate how such a ground state solution can be constructed in a simple WKB approximation.

Analyzing the form of the potential energy function, Eq. (6), one can draw several observations concerning the energy E of the ground state solution of Eq. (5): (i) there is a discrete spectrum which implies a functional relation between E and $\hat{\gamma}^2$, (ii) for $\hat{\gamma}^2 \rightarrow \infty$ the energy E also goes to ∞ , (iii) for $\hat{\gamma}^2 \rightarrow 0$ the energy E also goes to zero, (iv) for $E \gg 1/4$ the eigenstate is a function with a single maximum at $t = 0$, (v) for $E \ll 1/4$ the eigenstate is a function with two symmetrically located maxima [9]. The transition occurs near E equal to the peak value $U_{max} = 1/4$ that corresponds to $D_s = 1/2$. This explains the observation made in [4] that the solution changes its character when the Suydam parameter D_s is about twice its marginal stability value.

For the case $E \gg 1/4$ one can readily estimate the eigenvalue by the WKB method. Let t_0 be the turning point for a given value of E , see Fig. (1),

$$t_0 = \frac{1}{2} \ln E - \ln \hat{\gamma} \quad (8)$$

As the potential function is very steep one can neglect the potential energy between the turning points, like for a square-well potential, and write the WKB turning point condition

$$4\sqrt{E} \left(\frac{1}{2} \ln E - \ln \hat{\gamma} \right) = \pi \quad (9)$$

This gives the asymptotic relation

$$\hat{\gamma} \propto \sqrt{E} \quad (10)$$

Note that this asymptotic relation (rather obvious from the form of Eq. (3)) is not linear, as stated in [4], it looks linear only for relatively small values of t .

Now turning to the opposite limit $E \ll 1/4$. Eq. (5) can still be treated as a two turning point problem with turning points t_1 and t_2 , see Fig. (1),

$$\begin{aligned}
t_1 &= \ln \frac{1}{2} - \frac{1}{2} \ln E \\
t_2 &= \frac{1}{2} \ln E - \ln \hat{\gamma}
\end{aligned}
\tag{11}$$

Now again neglecting the potential energy between the turning points one can write the WKB condition as

$$2\sqrt{E}(t_2 - t_1) = \pi \tag{12}$$

Then the growth rate is

$$\hat{\gamma} = E \exp\left(-\frac{\pi/2}{\sqrt{E}}\right) \tag{13}$$

Near marginal stability the growth rate is exponentially small, as was established in [4], however our asymptotic relation Eq. (13) is slightly different from that derived in [4], $\hat{\gamma} \approx \exp(-\pi/\sqrt{E})$.

The double-peak form of the eigenfunction is an interesting feature. The distance between the peaks of the eigenfunction is

$$\Delta t \approx t_1 + t_2 \approx \ln(1/\sqrt{\hat{\gamma}}) \tag{14}$$

In the original space that corresponds to

$$\Delta x = \hat{\gamma} \sinh t \propto \sqrt{\hat{\gamma}}, \tag{15}$$

so the peaks are getting arbitrarily close as the marginal stability boundary is approached. That makes it difficult to capture the double-peak form, especially numerically, if the original spatial variable x is used.

In conclusion, we have shown a new simple way of deriving the Suydam criterion for stability of ideal, cylindrical, pressure-driven, localized, interchange modes, and calculating the growth rates of unstable modes. The character of solution changes markedly from the limit $D_s \gg 1/2$ where the eigenfunction has a single peak and $\hat{\gamma} \propto \sqrt{D_s}$, to the marginal stability limit $D_s \rightarrow 1/4$ where the eigenfunction has a double-peak form and the growth rate is exponentially small, $\hat{\gamma} = 2(D_s - 1/4) \exp(-\pi/2\sqrt{D_s - 1/4})$.

Acknowledgements

This work is performed for USDOE by Univ. Calif. LLNL under contract W-7405-ENG-48.

References

- [1] B. R. Suydam, in *Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 81, p.157.
- [2] J. P. Freidberg, “Ideal Magnetohydrodynamics”, Plenum Press (1987).
- [3] R. D. Hazeltine, J. D. Meiss “Plasma confinement”, Addison-Wesley Publishing Company (1992).
- [4] S. Gupta, J. D. Callen, and C. C. Hegna, *Phys. Plasmas* **9**, 3395, (2002).
- [5] R.M.Kulsrud, *Phys. Fluids* **6**, 904, (1963).
- [6] T. E. Stringer, *Nucl. Fusion* **15**, 125 (1975).
- [7] S. Yoshikawa and R.B. White, *Phys. Fluids* **23**(4), 791 (1980).
- [8] John D. Pryce, “Numerical Solution of Sturm-Liouville Problems”, Clarendon Press (1993).
- [9] L. D. Landau and E. M. Livshits, “Quantum Mechanics: non-relativistic theory”, Pergamon Press (1977).

Figure captions

Fig. 1. Shown schematically the form of the potential energy function $U(t)$ and the lowest eigenstates $v(t)$ for $E \gg 1/4$ and for $E \ll 1/4$.

