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# NOVEL TWO-SCALE DISCRETIZATION SCHEMES FOR LAGRANGIAN HYDRODYNAMICS

PANAYOT S. VASSILEVSKI

**ABSTRACT.** In this report we propose novel higher order conservative schemes of discontinuous Galerkin (or DG) type for the equations of gas dynamics in Lagrangian coordinates suitable for general unstructured finite element meshes. The novelty of our approach is in the formulation of two-scale non-oscillatory function recovery procedures utilizing integral moments of the quantities of interest (pressure and velocity). The integral moments are computed on a primary mesh (cells or zones) which defines our original scale that governs the accuracy of the schemes. In the non-oscillatory smooth function recovery procedures, we introduce a finer mesh which defines the second scale. Mathematically, the recovery can be formulated as nonlinear energy functional minimization subject to equality and nonlinear inequality constraints. The schemes are highly accurate due to both the embedded (local) mesh refinement features as well as the ability to utilize higher order integral moments. The new DG schemes seem to offer an alternative to currently used artificial viscosity techniques and limiters since the two-scale recovery procedures aim at resolving these issues. We report on some preliminary tests for the lowest order case, and outline some possible future research directions.

## 1. MOTIVATION AND BACKGROUND

In this report, we propose a novel general approach of constructing highly accurate and conservative schemes. The schemes are of discontinuous Galerkin type (cf., [5], [20], [26]) and are derived from integral form of the equations written on a given primal mesh (cells or volumes). To close the overall computational scheme we introduce procedures that provide non-oscillatory smooth function recovery from computed by the scheme integral moments. We achieve high accuracy by incorporating second (finer) scale of resolution in the non-oscillatory function recovery procedures, that can take into account the higher order integral moments. In this respect our schemes are related to the ENO (WENO) schemes ([18], [19], see also [36]-[37], [27]). The main difference is that we use local mesh refinement and utilize global energy functional minimization as well as incorporate constraints in the non-oscillatory smooth function recovery procedures. The constraints have physical meaning, namely non-negativity of the internal energy (or pressure).

Despite the long history and success in the area of Lagrangian hydrodynamics computations many of the currently used codes experience various difficulties (cf. [1]) that include: (1) handling of general grids and the frequently arising “hourglassing” problem (or mesh tangling) while moving the mesh; (2) the “art” of using artificial viscosity in the

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general unstructured mesh setting; (3) preserving symmetries; and (4) the energy conservation of the schemes. Our new schemes offer some feasible algorithmic directions that can address the challenges listed above. Some success has been achieved in the recent works in [30]–[31] and [34] where general finite element meshes were used in a number of difficult well-studied Lagrangian hydro test problems (such as Sedov test [33], Noh test [28], etc.) Our approach however has substantial differences. It combines the energy conservative finite volume (or FV) method with finite element based (equality and or inequality) constrained function recovery (on meshes with dynamic multilevel local refinement) to achieve higher accuracy on general polygonal (or polyhedral) meshes, the novelty being in the introduction of the energy minimization functionals in the recovery procedures. Thus, if efficiently implemented, the recovery procedures can augment the existing staggered grid finite volume codes which deal with piecewise constant quantities. In this respect our newly proposed schemes do not drastically deviate from the existing ones; they can be viewed as a natural extension of the more traditional finite volume schemes in a direction of improving accuracy (and maintaining conservation) with the potential of supplying efficient algorithms for removing the “hourglassing” and eliminating the need of artificial viscosity and limiters (cf., [7]–[8], [9], [12]) since the function recovery procedures that we propose are aimed explicitly at handling these issues. Another feature of practical importance of our schemes is that they are able to handle fairly general equations of state (or E.O.S.) as long as the E.O.S. take as input integral moments (of known quantities coming from the kinetic equations for example, or by other means such as tables, cf. [13]) and produce as output integral moments of the pressure.

In this report we focus on the construction of the schemes, their implementation and preliminary testing. A main task is the solution of the nonlinear constrained minimization problems coupled with the multilevel local refinement involved in the non-oscillatory smooth function recovery. The computational algorithmic challenges posed by our new approach are devising dynamic local refinement procedures, constructing efficient solution algorithms for the constrained nonlinear minimization problems with quadratic inequality constraints that arise in the the smooth function recovery procedures and they will be addressed in more details in a future study.

The remainder of the report is structured as follows. After some introduction to Lagrangian hydrodynamics, we describe our discretization strategy. Then, the function recovery procedures are presented. The overall computational scheme is summarized in Algorithm 3.1. The extension of the schemes to higher order integral moments is commented out in § 3.3. Some solution strategies for the constrained minimization problems are outlined in § 4. Our preliminary numerical tests are presented in § 5. In the final section § 6, we summarize our conclusions and outline a number of future research directions.

## 2. THE EQUATIONS IN EULERIAN AND LAGRANGIAN COORDINATES

The equations of Lagrangian gas dynamics are fundamental in many application areas. For an introduction on this topic and also on the construction of some more traditional finite volume schemes, we refer to [25] and [35]. The purpose of this section is to describe the equations and formulate them in a somewhat more general integral form that is the basis for the derivations of our high order conservative DG schemes.

The presentation in the following subsections is based on [17].

**2.1. The equation of gas dynamics in Eulerian coordinates.** The Euler equations for a compressible inviscid fluid (where the heat conduction is neglected) can be written in the following conservative form:

$$(2.1) \quad \begin{aligned} \frac{\partial \varrho}{\partial t} &= -\operatorname{div}(\varrho \mathbf{v}), \\ \frac{\partial(\varrho \mathbf{v})}{\partial t} &= -\nabla p - \sum_{j=1}^d \frac{\partial(\varrho v_j \mathbf{v})}{\partial x_j}, \\ \frac{\partial(\varrho E)}{\partial t} &= -\operatorname{div}((\varrho E + p)\mathbf{v}). \end{aligned}$$

Here,  $\varrho$  is the density of the fluid,  $\mathbf{v} = (v_1, \dots, v_d)$  is the fluid velocity,  $p$  is the pressure,  $e$  is the specific (per unit mass) internal energy, and  $E = e + \frac{1}{2} |\mathbf{v}|^2$  is the specific (per unit mass) total energy. The equations in (2.1) describe the laws of conservation of mass, momentum, and total energy of the fluid. There is one more equation, referred to as equation of state (or E.O.S., see [13]) associated with (2.1) that specifies the pressure  $p$ . In general it has the form

$$p = p(\varrho, e) = \operatorname{EOS}(\varrho, e),$$

which in the case of polytropic ideal gas reads  $p = (\gamma - 1) \varrho e$ , for a constant  $\gamma > 1$ .

In what follows, we do not assume any specific form of the E.O.S.; the schemes that we propose require that the E.O.S. takes as input integral moments of  $\varrho e$  and produces as output integral moments of the pressure with the property that the more accurate the input is (such as higher order moments) the more accurate the output will be (such as higher order moments of the pressure, perhaps on different scale). This is important if we want to extend our DG schemes to apply to the kinetic Boltzmann equation needed for developing coupled kinetic-hydrodynamic models (cf. [11], [10], [17], [29]).

**2.2. The equations of gas dynamics in Lagrangian coordinates and their integral form.** Here, we derive the equations of gas dynamics in Lagrangian coordinates in somewhat more general form than is traditionally used. This will serve as the basis for the derivation of our higher order DG schemes.

**2.2.1. Lagrangian coordinates.** Let  $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$  be the velocity field of the fluid flow. We consider the dynamical system

$$(2.2) \quad \frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t)$$

with initial condition

$$(2.3) \quad \mathbf{x}|_{t=0} = \boldsymbol{\xi}.$$

Consider the mapping  $\boldsymbol{\xi} \mapsto$  the solution  $\mathbf{x}(\boldsymbol{\xi}, t)$  of (2.2). By definition, the pair  $(\boldsymbol{\xi}, t)$  is called Lagrangian coordinates associated with the velocity field  $\mathbf{v}$ . If

$$(2.4) \quad J(\boldsymbol{\xi}, t) = \det \left( \frac{\partial x_i(\boldsymbol{\xi}, t)}{\partial \xi_j} \right),$$

then

$$\frac{\partial J(\boldsymbol{\xi}, t)}{\partial t} = J(\boldsymbol{\xi}, t) (\operatorname{div} \mathbf{v})(\mathbf{x}(\boldsymbol{\xi}, t), t).$$

Given a function  $\varphi(\mathbf{x}, t)$  in terms of Eulerian coordinates, we denote by  $\bar{\varphi} = \bar{\varphi}(\boldsymbol{\xi}, t)$  the same function in Lagrangian coordinates, i.e.,

$$\bar{\varphi}(\boldsymbol{\xi}, t) = \varphi(\mathbf{x}(\boldsymbol{\xi}, t), t).$$

It is clear then that the following identity holds

$$(2.5) \quad \frac{\partial(\bar{\varphi}J)}{\partial t} = J \left( \frac{\partial \bar{\varphi}}{\partial t} + \overline{\operatorname{div}(\varphi \mathbf{v})} \right).$$

2.2.2. *Integral form of the equations.* Consider first  $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$ . For a given multi-index  $\underline{\alpha} = (\alpha_i)_{i=1}^d$ , denote  $\mathbf{x}^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$ . For any given cell  $V = V(t)$ , we have the following integral form of the conservation of mass:

$$(2.6) \quad \begin{aligned} \frac{\partial}{\partial t} \int_{V(t)} \mathbf{x}^\alpha \rho \, d\mathbf{x} &= \frac{\partial}{\partial t} \int_{V(0)} \boldsymbol{\xi}^\alpha \bar{\rho} J \, d\boldsymbol{\xi} \\ &= \int_{V(0)} \boldsymbol{\xi}^\alpha \frac{\partial \bar{\rho} J}{\partial t} \, d\boldsymbol{\xi} \\ &= \int_{V(0)} \boldsymbol{\xi}^\alpha \overline{J \left( \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right)} \, d\boldsymbol{\xi} \\ &= \int_{V(t)} \mathbf{x}^\alpha \left( \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right) \, d\mathbf{x} \\ &= 0. \end{aligned}$$

For the integral form of the conservation of momentum equation, we obtain (using the formula (2.5) for  $\varphi = \rho \mathbf{v}$  and  $\mathbf{v} = (v_j)_{j=1}^d$ )

$$(2.7) \quad \begin{aligned} \frac{\partial}{\partial t} \int_{V(t)} \mathbf{x}^\alpha \rho \mathbf{v} \, d\mathbf{x} &= \frac{\partial}{\partial t} \int_{V(0)} \boldsymbol{\xi}^\alpha \overline{\rho \mathbf{v}} J \, d\boldsymbol{\xi} \\ &= \int_{V(0)} \boldsymbol{\xi}^\alpha \frac{\partial \overline{\rho \mathbf{v}} J}{\partial t} \, d\boldsymbol{\xi} \\ &= \int_{V(t)} \mathbf{x}^\alpha \left( \sum_{j=1}^d \frac{\partial \rho v_j \mathbf{v}}{\partial x_j} + \frac{\partial \rho \mathbf{v}}{\partial t} \right) \, d\mathbf{x} \\ &= - \int_{V(t)} \mathbf{x}^\alpha \nabla p \, d\mathbf{x}. \end{aligned}$$

For the energy conservation equation using again formula (2.5) now for  $\varphi = \rho E$ , we obtain

$$(2.8) \quad \begin{aligned} \frac{\partial}{\partial t} \int_{V(t)} \mathbf{x}^\alpha \rho E \, d\mathbf{x} &= \frac{\partial}{\partial t} \int_{V(0)} \boldsymbol{\xi}^\alpha \overline{E \rho} J \, d\boldsymbol{\xi} \\ &= \int_{V(0)} \boldsymbol{\xi}^\alpha \frac{\partial \overline{E \rho} J}{\partial t} \, d\boldsymbol{\xi} \\ &= \int_{V(t)} \mathbf{x}^\alpha \left[ \frac{\partial(\rho E)}{\partial t} + \operatorname{div}(\rho E \mathbf{v}) \right] \, d\mathbf{x} \\ &= - \int_{V(t)} \mathbf{x}^\alpha \operatorname{div}(p \mathbf{v}) \, d\mathbf{x}. \end{aligned}$$

For the internal energy  $e$ , we have the following integral form

$$(2.9) \quad \frac{\partial}{\partial t} \int_{V(t)} \mathbf{x}^\alpha \rho e \, d\mathbf{x} = - \int_{V(t)} \mathbf{x}^\alpha p \operatorname{div} \mathbf{v} \, d\mathbf{x}.$$

The more general formulas (with  $\underline{\alpha} \neq 0$ ) are the basis for deriving our higher order DG schemes.

### 3. THE MAIN CONSTRUCTION

In this section, we outline the main tasks involved in the construction of the proposed schemes; namely, we describe the equations that are used to compute main integral moments, the need of the smooth function recovery procedures and our approach to formulate them, and eventually end up with an overall computational scheme. Then, we comment on the extension of the schemes to utilize higher order moments.

**3.1. An outline of the proposed schemes.** We begin with a fairly straightforward derivation of finite volume equations (corresponding to the lowest order case  $\underline{\alpha} = 0$ ). The schemes utilizing higher order moments (corresponding to the case  $\underline{\alpha} \neq 0$ ) can be constructed in the same manner and are commented on at the end of the section.

The purpose of this derivation is to demonstrate why smooth function recovery is needed. We formulate our novel function recovery procedures as nonlinear function minimization subject to equality and nonlinear quadratic inequality constraints. Also, we demonstrate the need of dynamical local mesh refinement in the recovery process.

**3.1.1. Conservative finite volume schemes.** Consider the conservation of momentum equation

$$\frac{\partial}{\partial t} \int_{V(t)} \varrho \mathbf{v} \, d\mathbf{x} = - \int_{V(t)} \nabla p \, d\mathbf{x}.$$

Use time discretization

$$t_{n+1} = t_n + \Delta t.$$

Let  $V_n = V(t_n)$ . As an example, for an explicit scheme, we have

$$\frac{1}{\Delta t} \left( \int_{V_{n+1}} \varrho \mathbf{v} \, d\mathbf{x} - \int_{V_n} \varrho \mathbf{v} \, d\mathbf{x} \right) = - \int_{V_n} \nabla p_h \, d\mathbf{x}.$$

Here,  $p_h$  is a finite element approximation of  $p$  (to be specified).

Let  $m(V) = \int_V \varrho \, d\mathbf{x}$  be the mass associated with a cell  $V$ . From the conservation of mass equation

$$\frac{\partial}{\partial t} \int_{V(t)} \varrho \, d\mathbf{x} = 0,$$

we have that the mass is constant, i.e.,

$$m(V_n) = \int_{V(t_n)} \varrho \, d\mathbf{x} = \int_{V(t_{n+1})} \varrho \, d\mathbf{x} = m(V_{n+1}).$$

Approximating  $\varrho$  at  $t = t_n$  with  $m(V_n)/|V_n| = m(V)/|V_n|$ , we end up with the following FV scheme:

$$m(V) \frac{1}{\Delta t} \left( \frac{1}{|V_{n+1}|} \int_{V_{n+1}} \mathbf{v}_{n+1} d\mathbf{x} - \frac{1}{|V_n|} \int_{V_n} \mathbf{v}_n d\mathbf{x} \right) = - \int_{V_n} \nabla p_h d\mathbf{x}.$$

It is clear that we can compute the average values

$$\frac{1}{|V_{n+1}|} \int_{V_{n+1}} \mathbf{v}_{n+1} d\mathbf{x}$$

without knowing the actual (smooth) finite element approximation  $\mathbf{v}_h$  to  $\mathbf{v}_{n+1}$ .

Thus the problem of function recovery arises.

3.1.2. *Smooth function recovery from averages.* Given the average values

$$\frac{1}{|V_{n+1}|} \int_{V_{n+1}} \mathbf{v}_{n+1} d\mathbf{x}$$

construct a smooth function  $\mathbf{v}_h$  (that has the prescribed averages) to be used in the approximation of the conservation of energy equation

$$\frac{1}{\Delta t} \left( \int_{V_{n+1}} \varrho e d\mathbf{x} - \int_{V_n} \varrho e d\mathbf{x} \right) = - \int_{V_{n+1}} p_h \operatorname{div} \mathbf{v}_h d\mathbf{x}.$$

We formulate one possible function recovery procedure based on minimizing certain energy functional subject to some constraints. We choose the total variation (or TV) functional to minimize since it gives non-oscillatory recovery. An illustration of a TV constrained minimization procedure is shown in Figure 1; a discontinuous (piecewise constant) function is approximated on a locally refined mesh by an  $H^1$ -conforming finite element function.

We note that the choice of other functionals (see, e.g., [3]) will not change the principal steps of the overall construction of our discretization schemes.

We need a second finite element mesh  $\mathcal{T}_h$ , a refinement of the primal (FV or finite element) mesh  $\mathcal{T}_H$ . We stress upon the fact that the accuracy of the scheme is determined by the primal mesh  $\mathcal{T}_H$ .

The total variation function recovery reads: Find a finite element function  $\mathbf{v}_h$  with minimal total variation

$$\mathbf{J}_{TV}(\mathbf{v}_h) = \int_{\Omega} |\nabla \mathbf{v}_h| d\mathbf{x} \mapsto \min,$$

with prescribed integral moments for all  $V = V_{n+1} \in \mathcal{T}_H$

$$\int_V \varrho \mathbf{v}_h d\mathbf{x}.$$

In the case of lowest order (FV) schemes, the latter expression is simply  $\frac{m(V)}{|V|} \int_V \mathbf{v}_h d\mathbf{x}$ . In addition to the above (equality) constraints, we will impose some inequality constraints

that represent the non-negativity of the internal energy. To this end, consider the conservation of energy equation (for  $T \in \mathcal{T}_h$ ):

$$\int_{T_{n+1}} \varrho_{n+1} E_{n+1} d\mathbf{x} = \int_{T_n} \varrho_n E_n d\mathbf{x}_n - \Delta t \int_{\partial T_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n} d\sigma.$$

Since

$$E = e + \frac{1}{2} |\mathbf{v}|^2,$$

and from physical consideration (nonnegative internal energy), we have

$$0 \leq \left( \int_{T_{n+1}} \varrho e d\mathbf{x} = \right) \int_{T_n} \varrho_n E_n d\mathbf{x}_n - \Delta t \int_{\partial T_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n} d\sigma - \frac{1}{2} \int_{T_{n+1}} \varrho_{n+1} |\mathbf{v}_h|^2 d\mathbf{x}.$$

This is a quadratic inequality constraint for  $\mathbf{v}_h = \mathbf{v}_{n+1}$  imposed on any  $T = T_{n+1} \in \mathcal{T}_h$  for given  $\varrho_{n+1}$  and  $p_h$ .

A similar somewhat simpler problem can be formulated for  $p_h$ . Let  $S_h$  be a  $H^1$ -conforming finite element space associated with the finer mesh  $T_h$  (obtained by dynamic local refinement of the primal mesh  $\mathcal{T}_H$  at every time step). Find a finite element function  $p_h \in S_h$  such that

$$J_{TV}(p_h) = \int_{\Omega} |\nabla p_h| d\mathbf{x} \mapsto \min,$$

subject to the equality constraints (for all  $V \in \mathcal{T}_H$ ) using the E.O.S. (in its simple form):

$$\frac{1}{|V|} \int_V p_h d\mathbf{x} = \bar{p} \equiv \frac{\gamma - 1}{|V|} \int_V \varrho e d\mathbf{x}.$$

Note that the quadratic inequality constraint for  $\mathbf{v}_h$  implies that  $\bar{p} \geq 0$ .

The equality constraints can be imposed (approximately) as a penalty via the Rudin-Osher-Fatemi (or ROF) functional ([32]):

$$J_{ROF}(p_h) = \lambda \|p_h - \bar{p}\|_0^2 + \int_{\Omega} |\nabla p_h| d\mathbf{x} \mapsto \min.$$

Here,  $\lambda > 0$  is a large parameter. This functional is very popular in noise removal algorithms (see, [38]).

We can view the penalty version of the above recovery procedure as a ‘‘regularization’’. We have to construct a smooth recovery  $p_h$  based on given cell-averages  $\bar{p}$ . We need  $p_h$  to be smooth in order to use its gradient in the discretization of the conservation of momentum equation. Since numerical differentiation (as is well-known) is a ‘‘ill-posed’’ problem, we need some regularization to avoid (or minimize) possible spurious oscillations. The ROF-functional rewritten as

$$\|p_h - \bar{p}\|_0^2 + \epsilon \int_{\Omega} |\nabla p_h| d\mathbf{x},$$

for a small  $\epsilon$ , contains the non-linear TV-term that gives rise to the needed regularization.

Typically, the  $L_1$ -term involving  $\nabla p_h$  is modeled using a nonlinear elliptic form; namely, we can use the approximation, for a small (mesh dependent) parameter  $\delta > 0$ ,

$$|\nabla p_h| \approx \frac{|\nabla p_h|^2}{\sqrt{\delta + |\nabla p_h|^2}}.$$

Then, the resulting PDE to be solved for  $p_h$  is nonlinear elliptic and can be posed variationally as follows

$$\int_{\Omega} ((a(|\nabla p_h|)\nabla p_h \cdot \nabla \varphi + \lambda p_h \varphi) \, d\mathbf{x} = \lambda \int_{\Omega} \bar{p} \varphi \, d\mathbf{x} \quad \text{for all } \varphi \in S_h.$$

Here  $a(s) = 1/\sqrt{\delta + s^2}$ .

**3.2. Implementation of the function recovery based FV schemes.** In this section, we summarize in an algorithm form (presented for the lowest order case) the main steps needed to be implemented to test our new DG schemes.

We have a primal (moving) mesh  $\mathcal{T}_H$ . In the recovery procedures, we need a dynamically constructed mesh  $\mathcal{T}_h$  that is a refinement of  $\mathcal{T}_H$ . With  $\mathcal{T}_h$  we associate a finite element space  $S_h$  that is  $H^1$ -conforming. Its vector version will be denoted  $\mathbf{S}_h = (S_h)^d$ . A typical choice is  $S_h$  piecewise linear.

**Algorithm 3.1** (Conservative FV Scheme).

Let  $\{\mathbf{x}_n\}$  be the set of vertices of the primal cells in  $\mathcal{T}_H$  at time  $t_n$ . The algorithm below computes  $\mathbf{x}_h, \mathbf{v}_h \in \mathbf{S}_h$  and  $p_h \in S_h$  by iterations. It also computes the average values  $\bar{\mathbf{v}}$  and  $\bar{p}$  over the moved primal cells at time step  $t_{n+1}$ .

- To move the mesh, find a finite element function  $\mathbf{x}_h \in \mathbf{S}_h$  such that

$$\|\mathbf{x}_h - (\mathbf{x}_n + \Delta t \bar{\mathbf{v}}_n)\|_0^2 + \epsilon \int_{\Omega_n} |\nabla \mathbf{x}_h| \mapsto \min.$$

Then,  $\mathbf{x}_{n+1}$  equals  $\mathbf{x}_h$  restricted to the vertices of  $\mathcal{T}_H$  (at  $t = t_n$ ) and defines the vertices of the moved  $\mathcal{T}_H$  at time  $t = t_{n+1}$ . Thus, we can compute the volumes  $|V|$  for any cell  $V = V_{n+1} \in \mathcal{T}_H$ . Hence,

$$\varrho_{n+1} = \frac{m(V)}{|V_{n+1}|}, \quad \bar{\mathbf{v}}_{n+1} = \frac{1}{m(V)} \left[ \int_{V_n} \varrho_n \mathbf{v}_n \, d\mathbf{x}_n - \Delta t \int_{V_{n+1}} \nabla p_h \, d\mathbf{x}_{n+1} \right].$$

- Solve the constrained energy minimization problems for  $\mathbf{v}_h \in \mathbf{S}_h$  and  $p_h \in S_h$ :

$$\mathbf{J}_{ROF}(\mathbf{v}_h) = \|\mathbf{v}_h - \bar{\mathbf{v}}_{n+1}\|_{0, \varrho_{n+1}}^2 + \epsilon \int_{\Omega_{n+1}} |\nabla \mathbf{v}_h| \, d\mathbf{x}_{n+1} \mapsto \min,$$

$$J_{ROF}(p_h) = \|p_h - \bar{p}_{n+1}\|_0^2 + \epsilon \int_{\Omega_{n+1}} |\nabla p_h| \, d\mathbf{x}_{n+1} \mapsto \min,$$

subject to the quadratic inequality constraints for any  $T = T_{n+1} \in \mathcal{T}_h$

$$-\frac{1}{2} \int_{T_{n+1}} \varrho_{n+1} |\mathbf{v}_h|^2 d\mathbf{x} - \Delta t \int_{\partial T_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n} d\sigma + \int_{T_n} \varrho E_n d\mathbf{x}_n \geq 0.$$

The above algorithm contains several inner–outer loops. They can be arranged in a number of ways. For example, starting with some initial approximation for  $p_h$  (one possibility is to use an explicit time stepping approximation), we compute the average values  $\bar{\mathbf{v}}_{n+1}$  and then solve (approximately) the constrained minimization problem for  $\mathbf{v}_h$ . Once we have  $\mathbf{v}_h$  and  $p_h$ , we can compute the moments

$$\frac{1}{|V_{n+1}|} \int_{V_{n+1}} \varrho e d\mathbf{x} = \frac{1}{|V_{n+1}|} \left[ \int_{V_n} \varrho_n E_n d\mathbf{x}_n - \Delta t \int_{\partial V_{n+1}} p_h \mathbf{v}_h \cdot \mathbf{n} d\sigma - \frac{1}{2} \int_{V_{n+1}} \varrho_{n+1} |\mathbf{v}_h|^2 d\mathbf{x} \right].$$

From the E.O.S. we can then compute the averages  $\bar{p}_{n+1} = \frac{1}{|V_{n+1}|} \int_{V_{n+1}} p_h d\mathbf{x}$  and solve the ROF–minimization problem for a new pressure approximation  $p_h$ . The process can be repeated several times. At every step, we refine the mesh  $\mathcal{T}_h$  gradually to capture the possible large gradients of  $p_h$  (and  $\mathbf{v}_h$ ). The design of an efficient refinement strategy combined with the above non–linear iterations and its analysis is one of the main computational challenges and tasks involved in the construction of the proposed DG schemes that we will investigate in more detail in a future study.

We comment on the fact that in Algorithm 3.1 we have specified one possible way to move the mesh. We have chosen the TV regularization term  $\epsilon \int_{\Omega_n} |\nabla \mathbf{x}_h| d\mathbf{x}$  in order to prevent the mesh from tangling. Other approaches to avoid mesh tangling are found in [15], [16], see also [39], [40] and [2], or some postprocessing is generally applied (sometimes referred to as “hourglass” filtering).

Finally, it is worth noting that the described schemes are first order accurate in time. Higher order time stepping schemes are possible (e.g., Runge-Kutta), and this will be investigated in detail in a future study (especially, in the case of higher order moments).

**3.3. Extensions to higher order moments.** We now discuss our approach to incorporate higher order moments ( $\alpha \neq 0$ ) in the proposed DG schemes. In this case, we need to consider a better approximation than piecewise constant of the density  $\varrho$ . If an element  $T$  at time step  $t = t_n$  has moved to a position  $T_n$ , then the conservation of mass (see (2.6)) gives us the relations

$$\int_{T_{n+1}} \mathbf{x}^\alpha \varrho d\mathbf{x} = \int_{T_n} \mathbf{x}^\alpha \varrho d\mathbf{x} = \dots = \int_{T_0} \mathbf{x}^\alpha \varrho d\mathbf{x}.$$

At the initial time  $t = t_0$ ,  $\rho$  is given and hence all moments  $\int_{T_n} \mathbf{x}^\alpha \varrho d\mathbf{x}$  are in principle computable. To save storage, the needed moments can be computed whenever required by the scheme. Thus, we can assume that for any given polynomial  $\varphi(\mathbf{x})$ , the moments

$$\int_{T_n} \varphi(\mathbf{x}) \varrho d\mathbf{x}$$

are computationally available so they can be used to construct higher order (DG) schemes. In the recovery procedures, we can still use piecewise linear functions  $p_h$  and  $\mathbf{v}_h$ , hence expressions of the form

$$\int_{T_n} \varphi(\mathbf{x}) \varrho \mathbf{v}_h d\mathbf{x}, \int_{T_n} \varphi(\mathbf{x}) \varrho |\mathbf{v}_h|^2 d\mathbf{x}, \int_{T_n} \varphi(\mathbf{x}) \varrho p_h d\mathbf{x}, \text{ etc.}$$

for polynomials  $\varphi(\mathbf{x})$  (up to certain degree  $p$ ) are also computable (since  $\mathbf{v}_h$  and  $p_h$  are polynomials on  $T_n$ ). Therefore the approach outlined in the previous section is readily generalized in the higher order moments case. Alternatively, we may use local “p”-refinement (that is, having  $\underline{\alpha}$  vary over the primal cells), or combination of both, which are all viable options to be further investigated.

#### 4. COMPUTATIONAL ISSUES

As already mentioned, the nonlinear TV functional is not elliptic. In practice, we approximate it with the elliptic one:

$$|\nabla p_h| \approx \frac{1}{\sqrt{\delta^2 + |\nabla p_h|^2}} |\nabla p_h|^2,$$

for a mesh-dependent tolerance  $\delta$ . The approximation to the ROF functional gives rise to a quadratic (matrix-vector) functional

$$\mathcal{J}(\mathbf{v}) \equiv \frac{1}{2} \mathbf{v}^T (M + \epsilon A) \mathbf{v} - \mathbf{v}^T \mathbf{b} \mapsto \min,$$

where  $M$  is the mass-matrix and  $A$  corresponds to a Picard linearization of the non-linear elliptic bilinear form

$$a(u, \varphi) = \int_{\Omega} \frac{1}{\sqrt{\delta^2 + |\nabla u|^2}} \nabla u \cdot \nabla \varphi d\mathbf{x}.$$

The constrained minimization problem for the velocity is related to the contact or obstacle problems in mechanics (cf. [21], [22]). Traditionally, such constrained minimization problems with inequality constraints are solved based on monotone algorithms (i.e., at every iteration the objective functional is decreased). Some original references for multilevel algorithms for inequality constraints problems are [6], see also [23, 24]. In our case the new difficulty comes from the nonlinear (quadratic) inequality constraints which makes the construction of multilevel algorithms more challenging than having simple box inequality constraints since in the coarsening process we generally lose nestedness of the constraint sets. We note that an alternative to the monotone multigrid methods (cf., [23, 24]) can be based on the efficient interior point constrained minimization techniques (cf. [4]), which is left for a possible future study. For the present preliminary study, we chose the monotone Gauss-Seidel iterations. That is, for a current iterate  $\mathbf{v}$  we perform a loop over all indices  $i$  corresponding to the problem degrees of freedom. At every step  $i$ , based on the unit coordinate vector  $\mathbf{e}_i$ , we solve 1D quadratic minimization problem:

$$\mathcal{J}(\mathbf{v} + t\mathbf{e}_i) \mapsto \min, t \in \mathbb{R},$$

subject to the quadratic inequality constraints. The set of constraints provides a set of intervals where  $t \in \mathbb{R}$  can vary. Notice that all the intervals contain the origin. Thus the intersection of all intervals is non-empty. In summary, each 1D minimization step involves finding minimum of a (scalar) quadratic functional over a (scalar) interval. This ensures the monotonicity of the process. One monotone Gauss–Seidel loop is completed after all indices  $i$  are visited.

## 5. NUMERICAL ILLUSTRATION

We consider the following model test problem posed on the unit square domain  $\Omega$ . We set at the initial time  $t = 0$ ,  $p = 0$  outside a single volume (square)  $V \in \mathcal{T}_H$  and let  $p$  be equal to a constant on  $V$  such that the total energy satisfies  $\int_{\Omega} \rho E \, d\mathbf{x} = 1$ . Also,  $\mathbf{v} = 0$  and  $\rho = 1$  initially. We keep  $\mathbf{v} \cdot \mathbf{n} = 0$  on  $\partial\Omega$  for  $t \geq 0$  so that the domain  $\Omega$  stays fixed.

We show two examples. The first one corresponds to a coarse primal mesh, see. Fig. 1, 2 and 3. The remaining figures illustrate a second test performed on somewhat more realistic finer primal mesh. Both tests show conversion of internal energy into kinetic and vice-versa. In the 2nd test, the shock wave travels from the bottom left corner of  $\Omega$ , reaches the opposite one and starts coming back. Although the results are not as accurate, the potential of the schemes is clearly seen. We expect much better results when the higher order moments are incorporated combined with higher order time discretization.

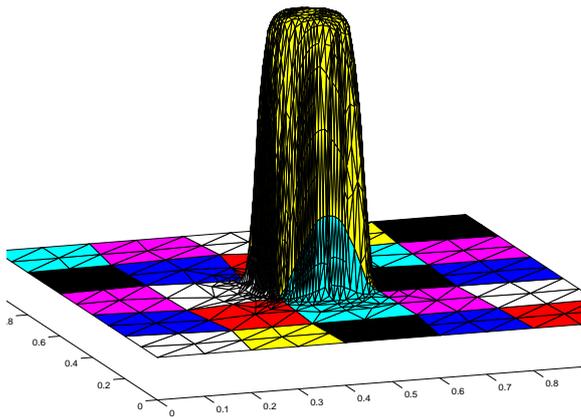
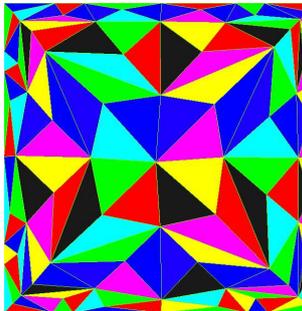
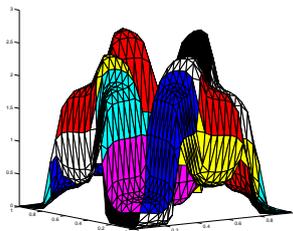
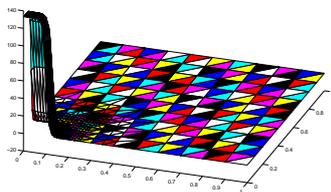
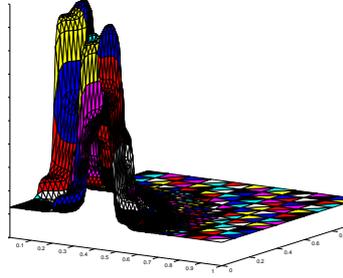
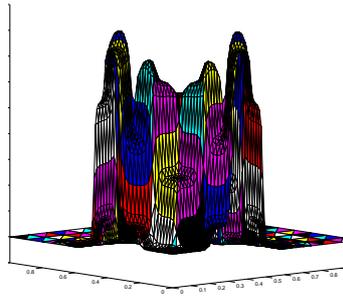
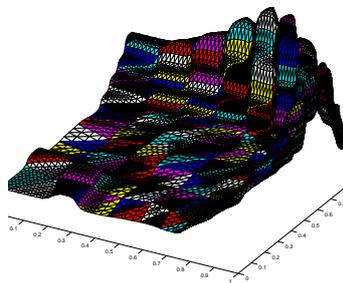


FIGURE 1. TV recovery of discontinuous pressure.

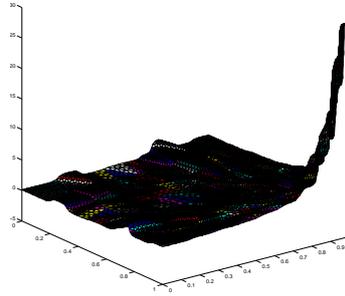
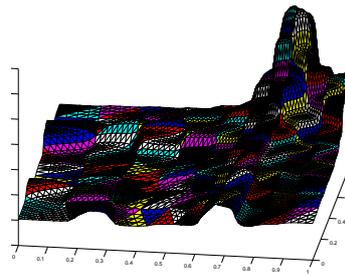
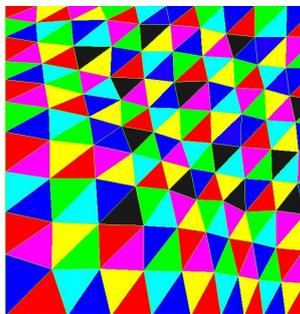
FIGURE 2. Moved mesh at time  $t = 0.0787$ .FIGURE 3. Recovered pressure at time  $t = 0.221$ .FIGURE 4. Recovered pressure at time  $t = 0.0005$ .

## 6. CONCLUSIONS

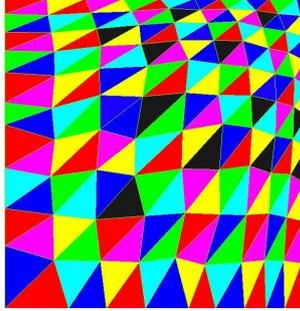
We have proposed new conservative finite volume schemes (for Lagrangian hydrodynamics). They are based on standard integral form of the conservation laws and utilize

FIGURE 5. Recovered pressure at time  $t = 0.2005$ .FIGURE 6. Recovered pressure at time  $t = 0.2005$ .FIGURE 7. Recovered pressure at time  $t = 0.8005$ .

non-oscillatory (TV based) function recovery. The function recovery procedures seem to be able to replace traditionally used “artificial viscosity” and limiters.

FIGURE 8. Recovered pressure at time  $t = 1.0005$ .FIGURE 9. Recovered pressure at time  $t = 1.1005$ .FIGURE 10. Moved mesh at time  $t = 0.4005$ .

The local mesh refinement used in the function recovery is essential and needs further study for efficiency.

FIGURE 11. Moved mesh at time  $t = 0.8005$ .

The most expensive part in the computation is the constrained minimization with quadratic inequality constraints. To speed it up, we need a multilevel procedure (not as straightforward due to the quadratic inequalities).

The monotone Gauss–Seidel in the pressure recovery has provable mesh–independent convergence (there are no inequalities).

A general observation from the presented numerical experiments is that the extension to higher order integral moments combined with higher order time discretization is essential which will be the topic of our future research.

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