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Erratum: ``Orbital Advection by Interpolation: A Fast and Accurate Numerical Scheme for Super-Fast MHD Flows'' (ApJ, 177, 373 [2008])

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Erratum: "Orbital Advection by Interpolation: A Fast and Accurate Numerical Scheme for Super-Fast MHD Flows" (ApJ, 177, 373 [2008])

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The descriptions of some of the numerical tests in our original paper are incomplete, making reproduction of the results difficult. We provide the missing details here. The relevant tests are described in §4 of the original paper (Figures 8-11).

We use the analytical solutions outlined by Johnson (ApJ, 660, 1375 [2007]) as the initial conditions for these tests. The incompressive solution is given by the real parts of expressions (80)-(82) of that paper. For imaginary ω and $\tilde{\omega}$ and a Keplerian rotation profile, these are

$$\delta \boldsymbol{v} = \tilde{\delta \boldsymbol{v}} \cos\left(\boldsymbol{k}\cdot\boldsymbol{x} + \frac{\pi}{4}\right) \tag{1}$$

and

$$\delta \boldsymbol{v}_A = \delta\tilde{\boldsymbol{v}}_A \cos\left(\boldsymbol{k}\cdot\boldsymbol{x} - \frac{\pi}{4}\right), \tag{2}$$

with

$$\tilde{\delta \boldsymbol{v}} = \mathcal{A}_i \left(k_x^2 - k^2, k_x k_y - \frac{k^2}{2\alpha}, k_x k_z + \frac{k^2 k_y}{2\alpha k_z}\right) \tag{3}$$

and

$$\delta\tilde{\boldsymbol{v}}_A = -\frac{\boldsymbol{v}_A\cdot\boldsymbol{k}}{|\omega|}\mathcal{A}_i \left(k_x^2 - k^2, k_x k_y + 2\alpha k_z^2, k_x k_z - 2\alpha k_y k_z\right), \tag{4}$$

where

$$\mathcal{A}_i = \epsilon c_s H \frac{|\tilde{\omega}|}{\Omega}\sqrt{\frac{|\omega|\Omega}{2|\tilde{\omega}^2|k^2 + \Omega^2 k_z^2}} \tag{5}$$

and

$$\alpha = \frac{\Omega|\omega|}{|\tilde{\omega}^2|}. \tag{6}$$

Here $H = c_s/\Omega$ is the disk scale height, $\boldsymbol{k}$ is the initial wave number of the perturbation and ϵ is an arbitrary perturbation amplitude; other symbols have their usual meanings. These

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solutions have been normalized to the correct dimensional units.[2] The density perturbation is given by

$$\frac{\delta\rho}{\rho_0} = \left(-\frac{\boldsymbol{v}_A}{c_s}\cdot\frac{\tilde{\delta\boldsymbol{v}}_A}{c_s} + \frac{2\Omega}{c_s k}\left[\frac{k_x}{k}\hat{\boldsymbol{y}} + \frac{k_y}{2k}\hat{\boldsymbol{x}}\right]\cdot\frac{\tilde{\delta\boldsymbol{v}}}{c_s}\right)\cos\left(\boldsymbol{k}\cdot\boldsymbol{x} - \frac{\pi}{4}\right). \tag{7}$$

The unstable branch of the incompressive dispersion relation is

$$|\tilde{\omega}^2| = \left(\frac{k_z\Omega}{k}\right)^2\left(\sqrt{1 + \left[\frac{4k\boldsymbol{v}_A\cdot\boldsymbol{k}}{k_z\Omega}\right]^2} - 1\right) \tag{8}$$

and

$$|\omega| = \sqrt{|\tilde{\omega}^2| - (\boldsymbol{v}_A\cdot\boldsymbol{k})^2}. \tag{9}$$

For our choice of initial parameters, $\boldsymbol{v}_A = \sqrt{15/16}(\Omega/k_z)\hat{\boldsymbol{z}}$ and $H\boldsymbol{k} = 2\pi(-2/10, 1/10, 1)$, these become

$$|\tilde{\omega}^2| = \Omega^2\frac{5}{21}\left(\sqrt{67} - 2\right) \simeq 1.47\Omega^2 \tag{10}$$

and

$$|\omega| = \Omega\sqrt{\frac{5}{21}}\left(\sqrt{67} - \frac{95}{16}\right)^{1/2} \simeq 0.732\Omega. \tag{11}$$

The perturbations in this limit are given by

$$\tilde{\delta\boldsymbol{v}} = -\frac{\mathcal{A}_i}{H^2}(2\pi)^2\left(\frac{101}{100}, \frac{1}{50} + \frac{21}{40\alpha}, \frac{1}{5} - \frac{21}{400\alpha}\right) \tag{12}$$

and

$$\tilde{\delta\boldsymbol{v}}_A = \sqrt{\frac{15}{16}}\frac{\Omega}{|\omega|}\frac{\mathcal{A}_i}{H^2}(2\pi)^2\left(\frac{101}{100}, \frac{1}{50} - 2\alpha, \frac{1}{5} + \frac{\alpha}{5}\right), \tag{13}$$

with

$$\frac{\mathcal{A}_i}{H^2} = \epsilon c_s\frac{|\omega|}{2\pi\Omega}\left(\frac{2}{\alpha\sqrt{67}}\right)^{1/2} \tag{14}$$

and

$$\alpha = \frac{\sqrt{21}\left(\sqrt{67} - 95/16\right)^{1/2}}{\sqrt{5}\left(\sqrt{67} - 2\right)} \simeq 0.497. \tag{15}$$

Dividing through by an overall factor of $H^2(k_x^2 - k^2) = -(2\pi)^2(101/100)$ gives the initial conditions quoted in our original paper (with $\epsilon = 10^{-6}$).

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[2]We set $4\pi\rho_0 = 1$ in our original paper, so that the magnetic field is equivalent to the Alfvèn velocity.

We make comparisons based upon the amplitude of the solution, i.e. $\tilde{\delta \boldsymbol{v}}$ and $\tilde{\delta \boldsymbol{v}_A}$ rather than $\delta \boldsymbol{v}$ and $\delta \boldsymbol{v}_A$. In Figures 8-10 of our original paper, then, the quantity that is being plotted is $\tilde{\delta \boldsymbol{v}}^2 + \tilde{\delta \boldsymbol{v}_A}^2$. To extract these quantities from the code we calculate departures from the mean, multiply by the appropriate cosine factor in expressions (1) and (2) (where $\boldsymbol{k}$ in this case is the time-dependent wave number) and integrate over the computational domain. This is done at each time step to give a time history of the perturbation amplitudes, which are then compared to the analytical solution. As a concrete example,

$$\left(\tilde{\delta v_x}[t]\right)_{numerical} = \frac{2}{L_x L_y L_z} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} vx^n_{ijk} \cos\left(\boldsymbol{k}[t^n] \cdot \boldsymbol{x}_{ijk} + \frac{\pi}{4}\right), \tag{16}$$

where vx^n_{ijk} is the radial velocity component at grid location (i, j, k) and time step n. Since the solution as expressed above breaks down as ω transitions from imaginary to real, we calculate the analytical amplitudes for the incompressive tests based upon an integration of the full set of linear equations.

The compressive solution is given by the real part of expressions (83)-(85) of Johnson (ApJ, 660, 1375 [2007]):

$$(\delta \boldsymbol{v}, \delta \boldsymbol{v}_A, \delta\rho) = \left(\tilde{\delta \boldsymbol{v}}, \tilde{\delta \boldsymbol{v}_A}, \tilde{\delta\rho}\right) \cos\left(\boldsymbol{k} \cdot \boldsymbol{x}\right), \tag{17}$$

with

$$\tilde{\delta \boldsymbol{v}} = \frac{\omega}{\tilde{\omega}} \mathcal{A}_c \left(\frac{\omega^2}{k^2}\boldsymbol{k} - \boldsymbol{v}_A \cdot \boldsymbol{k}\, \boldsymbol{v}_A\right), \tag{18}$$

$$\tilde{\delta \boldsymbol{v}_A} = \frac{\omega^2}{\tilde{\omega}} \mathcal{A}_c \left(\boldsymbol{v}_A - \frac{\boldsymbol{v}_A \cdot \boldsymbol{k}}{k^2}\, \boldsymbol{k}\right), \tag{19}$$

and

$$\frac{\tilde{\delta\rho}}{\rho_0} = \tilde{\omega}\mathcal{A}_c, \tag{20}$$

where

$$\mathcal{A}_c = \epsilon H k \sqrt{\frac{\omega\Omega}{\omega^4 - \left(\boldsymbol{v}_A \cdot \boldsymbol{k}\right)^2 c_s^2 k^2}}. \tag{21}$$

Our choice of initial parameters for this test, $\boldsymbol{v}_A = c_s(0.1, 0.2, 0.0)$ and $H\boldsymbol{k} = 4\pi(-2, 1, 1)$, gives $\boldsymbol{v}_A \cdot \boldsymbol{k} = 0$, so that the nonzero solution to the compressive dispersion relation is

$$\omega^2 = \left(c_s^2 + v_A^2\right) k^2. \tag{22}$$

The perturbations in this limit are given by

$$\left(\tilde{\delta \boldsymbol{v}}, \tilde{\delta \boldsymbol{v}_A}, \tilde{\delta\rho}/\rho_0\right) = \left(v_A\sqrt{1+\beta}\,\hat{\boldsymbol{k}}, \boldsymbol{v}_A, 1\right) \left(Hk\sqrt{\frac{\beta}{1+\beta}}\right)^{1/2}, \tag{23}$$

where $\beta = c_s/v_A$. For our initial conditions ($\beta = 20$), this is

$$\left(\tilde{\delta \boldsymbol{v}}, \tilde{\delta \boldsymbol{v}}_A, \tilde{\delta \rho}/\rho_0\right) = \left(\frac{c_s}{2}\sqrt{\frac{7}{10}}\frac{H\boldsymbol{k}}{4\pi}, \boldsymbol{v}_A, 1\right)\left(8\pi\sqrt{\frac{10}{7}}\right)^{1/2}, \tag{24}$$

which matches the numbers given in our original paper with $\epsilon = 10^{-6}$.

Figure 11 of our original paper shows the evolution of the azimuthal component of $\tilde{\delta \boldsymbol{v}}_A$. The numerical results are

$$\left(\tilde{\delta v}_{Ay}[t]\right)_{numerical} = \frac{2}{L_x L_y L_z}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sum_{k=1}^{N_z}\left(\frac{by^n_{ijk}}{\sqrt{4\pi\rho_0}} - v_{Ay}[t^n]\right)\cos\left(\boldsymbol{k}[t^n]\cdot\boldsymbol{x}_{ijk}\right). \tag{25}$$

and the analytical results are calculated within the code using the time dependent version of expression (24), i.e.

$$\left(\tilde{\delta v}_{Ay}[t]\right)_{analytical} = v_{Ay}[t^n]\left(8\pi\sqrt{\frac{10}{7}}\right)^{1/2}\cos\left(\sum_{n'=0}^{n}\omega[t^{n'}]\,dt^{n'}\right), \tag{26}$$

with $dt^0 = 0$.

As a final practical consideration, implementing the solutions as described above can introduce divergence into the initial conditions. To avoid this, we calculate the vector potential in the Coulomb guage ($\boldsymbol{k}\cdot\delta\boldsymbol{A} = 0$) for the above solutions and numerically calculate its curl to obtain the initial magnetic field perturbation. The perturbed vector potential is

$$\frac{\delta\boldsymbol{A}}{\sqrt{4\pi\rho_0}} = -\frac{\boldsymbol{v}_A\cdot\boldsymbol{k}}{|\omega|}\mathcal{A}_i k_z\left(2\alpha\left[\frac{k_x^2}{k^2}-1\right], 2\alpha\frac{k_x k_y}{k^2}-1, 2\alpha\frac{k_x k_z}{k^2}+\frac{k_y}{k_z}\right)\cos\left(\boldsymbol{k}\cdot\boldsymbol{x}+\frac{\pi}{4}\right) \tag{27}$$

for the incompressive solution and

$$\frac{\delta\boldsymbol{A}}{\sqrt{4\pi\rho_0}} = \frac{\omega^2\boldsymbol{v}_A\times\boldsymbol{k}}{\tilde{\omega}k^2}\mathcal{A}_c\sin\left(\boldsymbol{k}\cdot\boldsymbol{x}\right) \tag{28}$$

for the compressive solution. For our initial conditions, these reduce to

$$\frac{\delta\boldsymbol{A}}{\sqrt{4\pi\rho_0}} = \frac{\epsilon c_s H}{14}\left(\frac{1}{30\alpha\sqrt{67}}\right)^{1/2}\left(202\alpha, 4\alpha+105, 40\alpha-\frac{21}{2}\right)\cos\left(\boldsymbol{k}\cdot\boldsymbol{x}+\frac{\pi}{4}\right) \tag{29}$$

for the incompressive solution (with α given by expression [15]), and

$$\frac{\delta\boldsymbol{A}}{\sqrt{4\pi\rho_0}} = \epsilon\frac{c_s H}{60}\left(\frac{1}{\pi}\sqrt{\frac{5}{14}}\right)^{1/2}(2,-1,5)\sin\left(\boldsymbol{k}\cdot\boldsymbol{x}\right) \tag{30}$$

for the compressive solution.

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