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Accuracy of Reduced and Extended Thin-Wire Kernels

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Accuracy of Reduced and Extended Thin-Wire Kernels*

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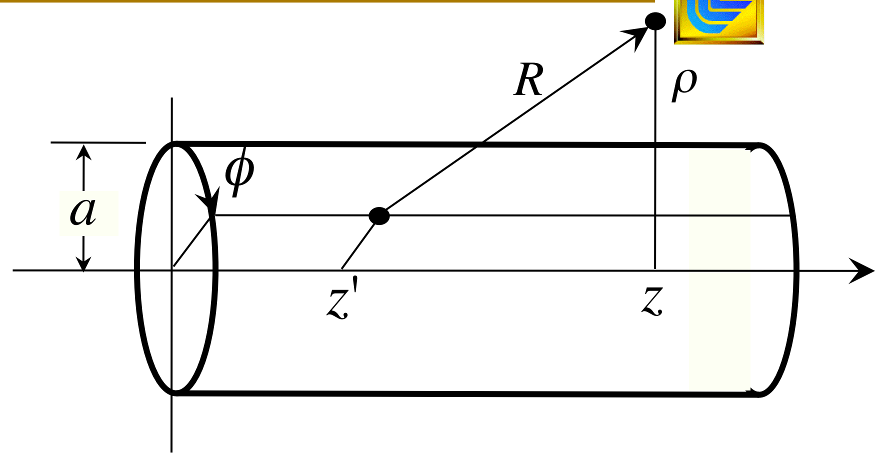
Several approximations of the thin-wire kernel have been used

$$\Phi(\rho,z) = \frac{j}{4\pi\omega\varepsilon}\int_{-\Delta/2}^{\Delta/2}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}\,d\phi\,dz'$$



- Reduced kernel (RTWK):
  current filament on the surface,
  evaluation on the axis (NEC-4)

- Exact kernel (Werner 1993, Wilton, Champagne 2006 ...)

- Extended kernel: integrate 1/R accurately, remainder RTWK (Hwu, Wilton 1988)

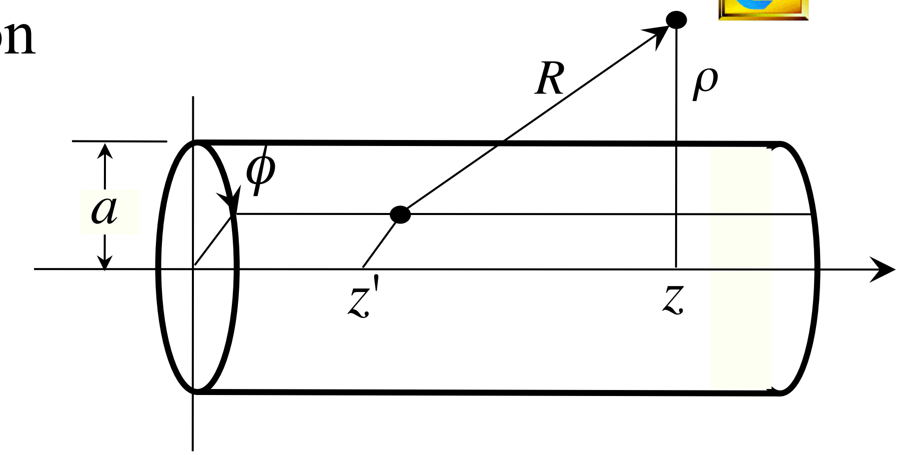
- Reduced kernel with wire end caps (Popovic)

Kernels involve different approximations of the surface integral

- Exact kernel: the Green's function is integrated over z and ϕ

$$R = \sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho\cos\phi}$$

$$\Phi(\rho,z) = \frac{j}{4\pi\omega\varepsilon}\int_{-\Delta/2}^{\Delta/2}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}\,d\phi\,dz'$$



- The reduced kernel (RTWK) uses a current filament at $\phi = \pi/2$

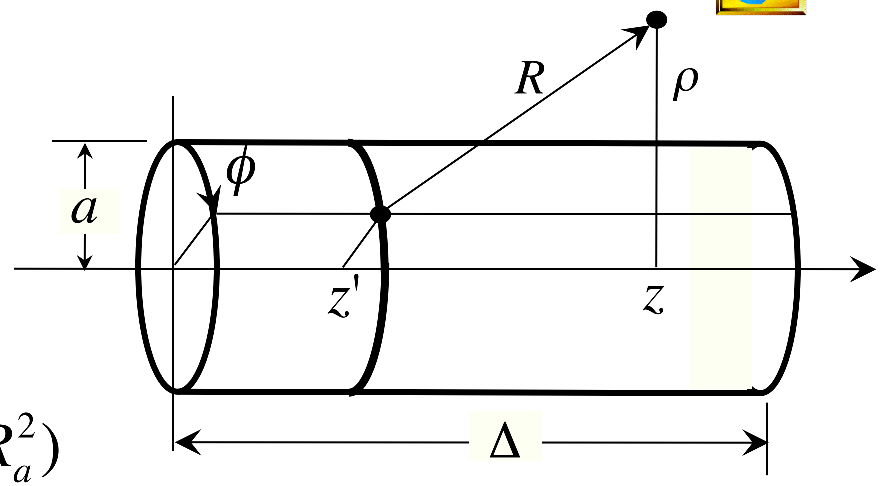
$$\Phi_0(\rho,z) = \frac{j}{2\omega\varepsilon}\int_{-\Delta/2}^{\Delta/2}\frac{e^{-jkR0}}{R_0}\,dz' \qquad R_0 = \sqrt{\rho^2 + a^2 + (z - z')^2}$$

Evaluation on segment axis

- The extended kernel (ETWK) integrates 1/R accurately

$$\Phi_1(\rho,z) = \frac{j}{4\pi\omega\varepsilon}\left(\overbrace{2\pi\int_{-\Delta/2}^{\Delta/2}\frac{e^{-jkR_0} - 1}{R_0}\,dz'}^{\text{RTWK approx.}} + \overbrace{\int_{-\Delta/2}^{\Delta/2}\int_{-\pi}^{\pi}\frac{1}{R}\,d\phi\,dz'}^{\text{Accurate double integral}}\right)$$

The 1/R term can be integrated over ϕ, then numerically in z’



$$R = \sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho\cos\phi}$$

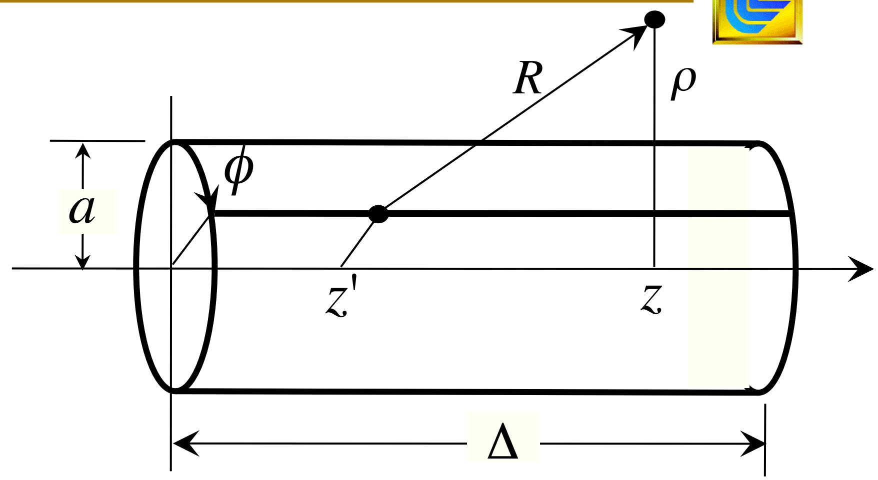
$$K_\phi(\rho, z, \phi) = \int_{-\pi}^{\pi} \frac{1}{R} d\phi = 4\,\frac{K(4a\rho / R_a^2)}{R_a}$$

$$R_a = \sqrt{(z - z')^2 + (a + \rho)^2}$$

K = Complete elliptic integral of first kind

- Integrate numerically over z’ and extract near the singularity

Or 1/R can be integrated over z', then numerically over φ



$$R = \sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho\cos\phi}$$

$$K_z(\rho, z, \phi) = \int_{-\Delta/2}^{\Delta/2} \frac{1}{R} dz' = \pm \log\left(\frac{z_2 \pm \sqrt{a^2 + z_2^2 + \rho^2 - 2a\rho\cos\phi}}{z_1 \pm \sqrt{a^2 + z_1^2 + \rho^2 - 2a\rho\cos\phi}} \right)$$

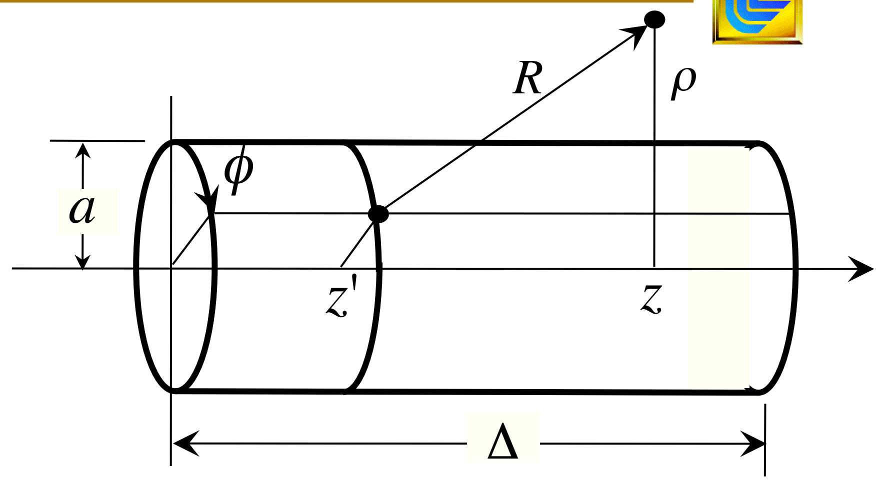
$$z_1 = -\Delta/2 - z, \qquad z_2 = \Delta/2 - z$$

Sign: $\begin{cases} + \text{ for } z_1, z_2 > 0 \\ - \text{ for } z_1, z_2 < 0 \end{cases}$

In the log form the singularity can be integrated exactly

$$R = \sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho\cos\phi}$$

$$z_1 = -\Delta/2 - z, \qquad z_2 = \Delta/2 - z$$



when -Δ/2< z < Δ/2 :

$$K_z(\rho,z,\phi) = \int_{-\Delta/2}^{\Delta/2} \frac{1}{R} dz'$$

$$= \log\left[\left(-z_1 + \sqrt{a^2 + z_1^2 + \rho^2 - 2a\rho\cos\phi}\right)\left(z_2 + \sqrt{a^2 + z_2^2 + \rho^2 - 2a\rho\cos\phi}\right)\right]$$

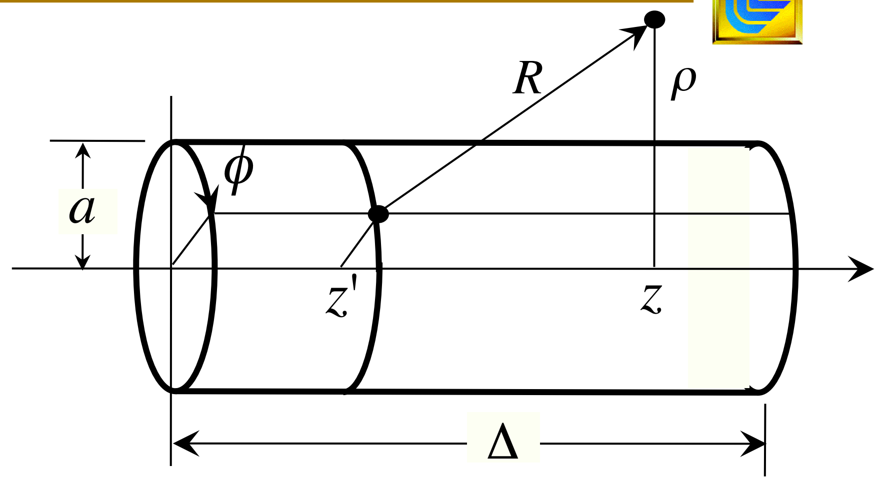
$$-\log\left(a^2 + \rho^2 - 2a\rho\cos\phi\right)$$

The last term, which may be singular, can be integrated over ϕ as

$$\int_{-\pi}^{\pi} \log\left(a^2 + \rho^2 - 2a\rho\cos\phi\right) d\phi = 4\pi\log\left[\max\left(a,\rho\right)\right]$$

Except when the evaluation point is near segment end, then extract.

The linear term (z'-z)/R can be integrated exactly



$$R = \sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho\cos\phi}$$

$$K(z) = \int_{-\pi}^{\pi}\int_{-\Delta/2}^{\Delta/2} \frac{z' - z}{R}\, dz'\, d\phi = \int_{-\pi}^{\pi}\left(R\big|_{z=\Delta/2} - R\big|_{z=-\Delta/2} \right) d\phi$$

$$= 4\left[\sqrt{z_2^2 + (a-\rho)^2}\, E\left(\frac{-4a\rho}{z_2^2 + (a-\rho)^2} \right) - \sqrt{z_1^2 + (a-\rho)^2}\, E\left(\frac{-4a\rho}{z_1^2 + (a-\rho)^2} \right) \right]$$

$$z_1 = -\Delta/2 - z, \qquad z_2 = \Delta/2 - z$$

E(x) = complete elliptic integral of the second kind

RTWK and ETWK errors were evaluated relative to the exact kernel

• Exact kernel evaluated with Mathematica’s NIntegrate function including singular integrals.
It needs a little help: MaxRecursion->20, PrecisionGoal->10

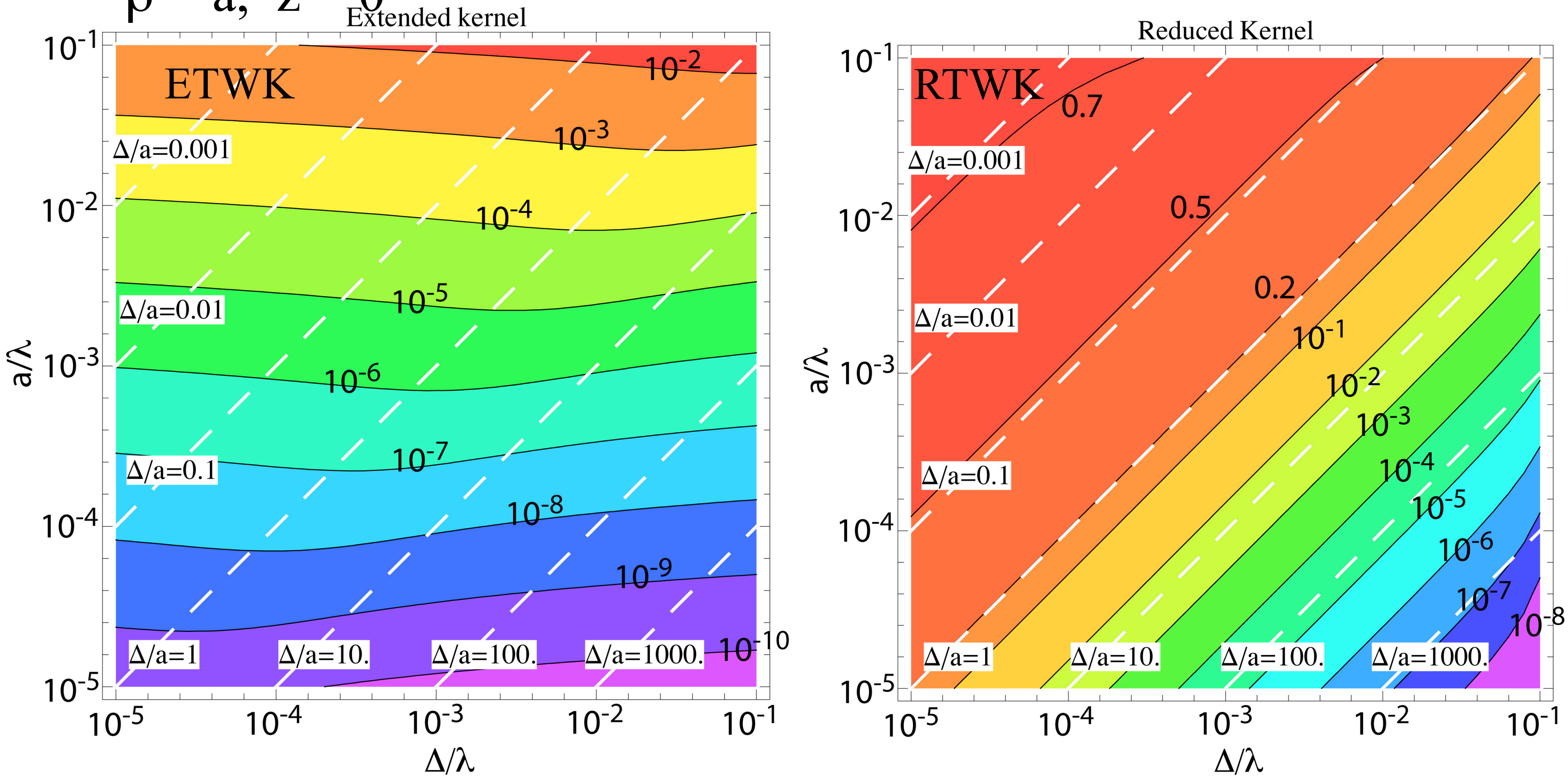
ETWK: log form with ϕ integral done with NIntegrate

• “Error” in individual matrix elements may not reflect error in the solution.

Self-term error relative to the exact kernel:

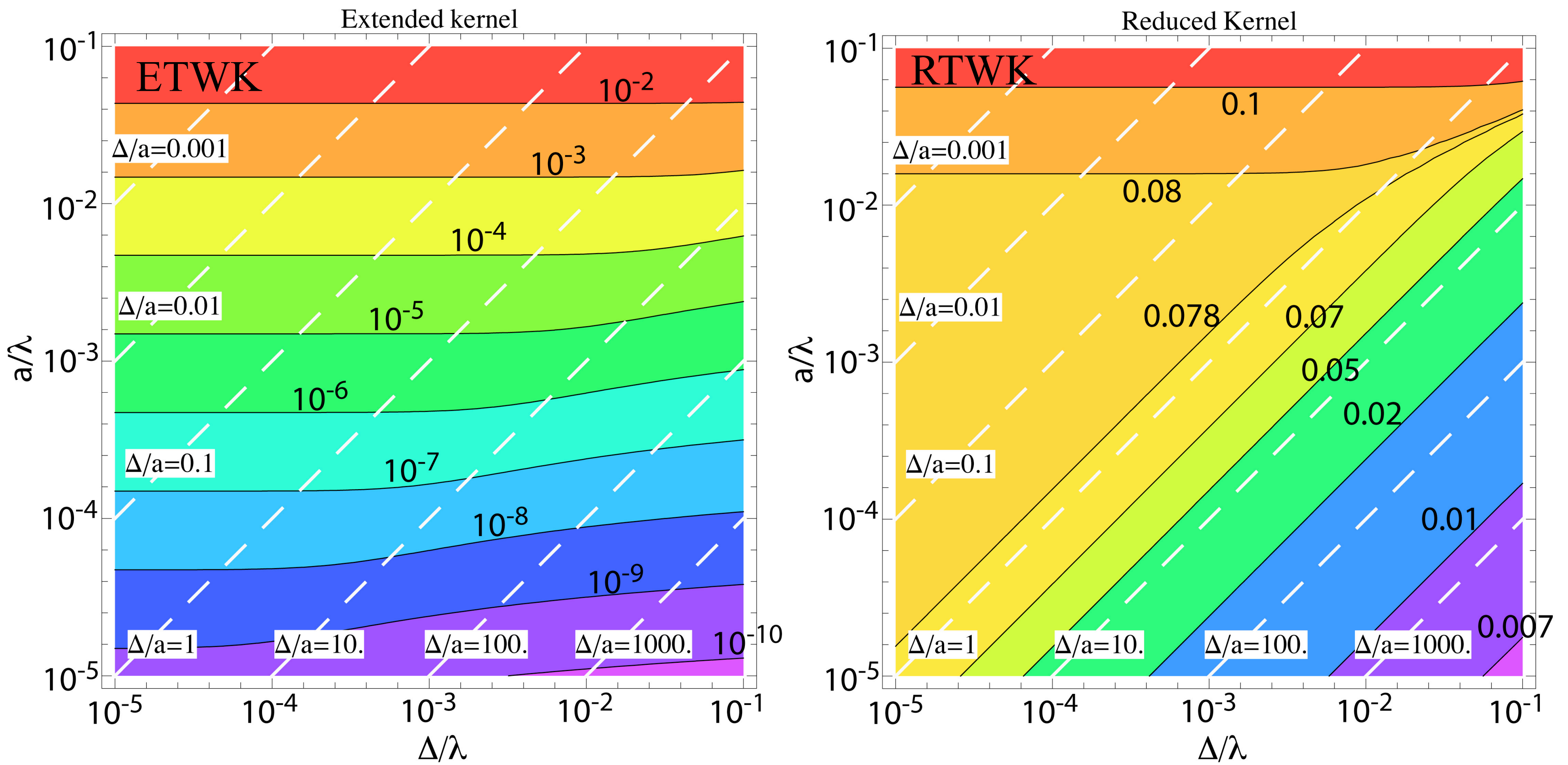
- RTWK error depends on Δ/a
- ETWK is limited by phase error across the wire

$\rho = a, \ z = 0$



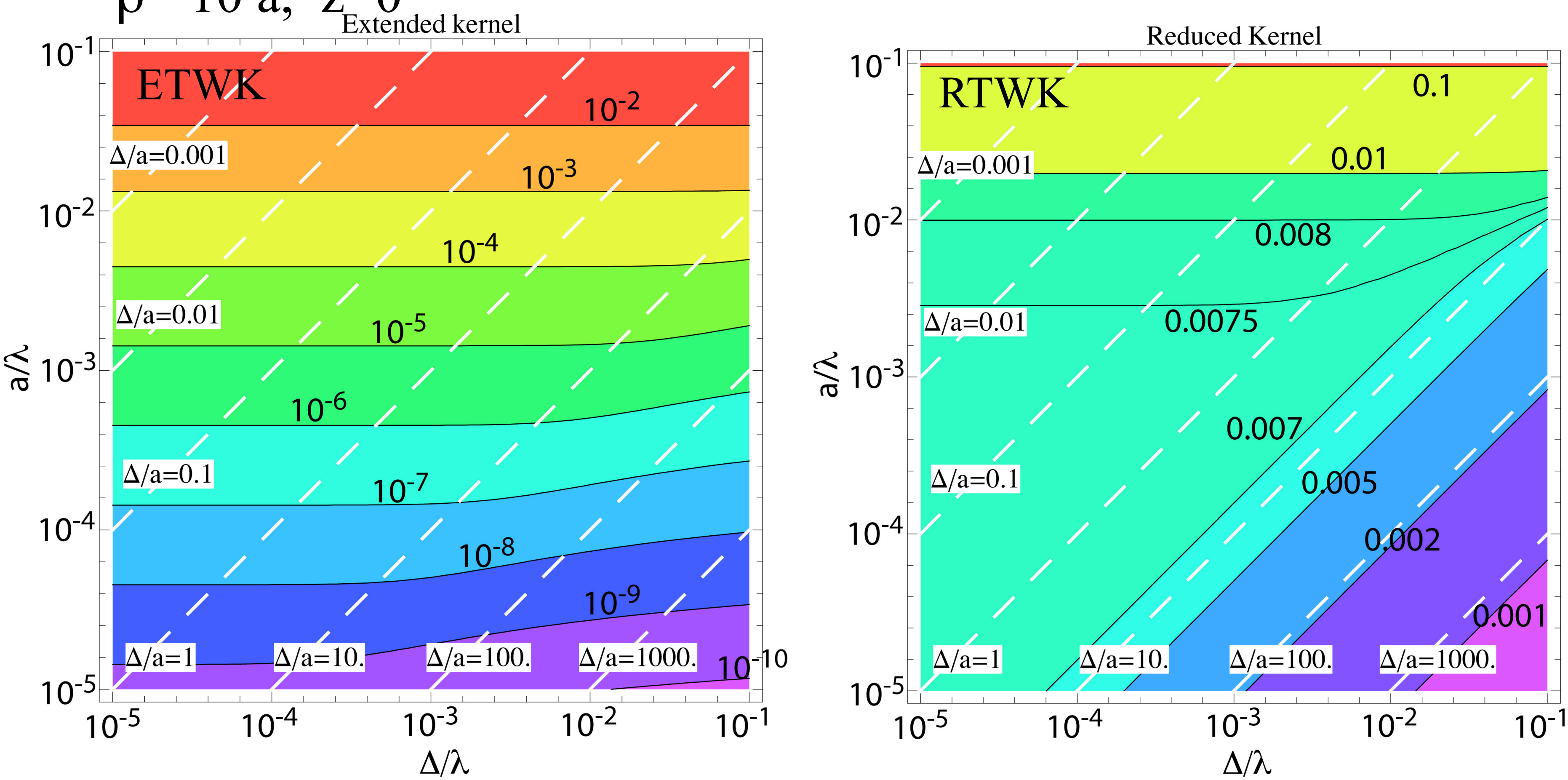
With ρ =3 a the RTWK error is larger for small a

$\rho = 3\ a,\ \ z = 0$



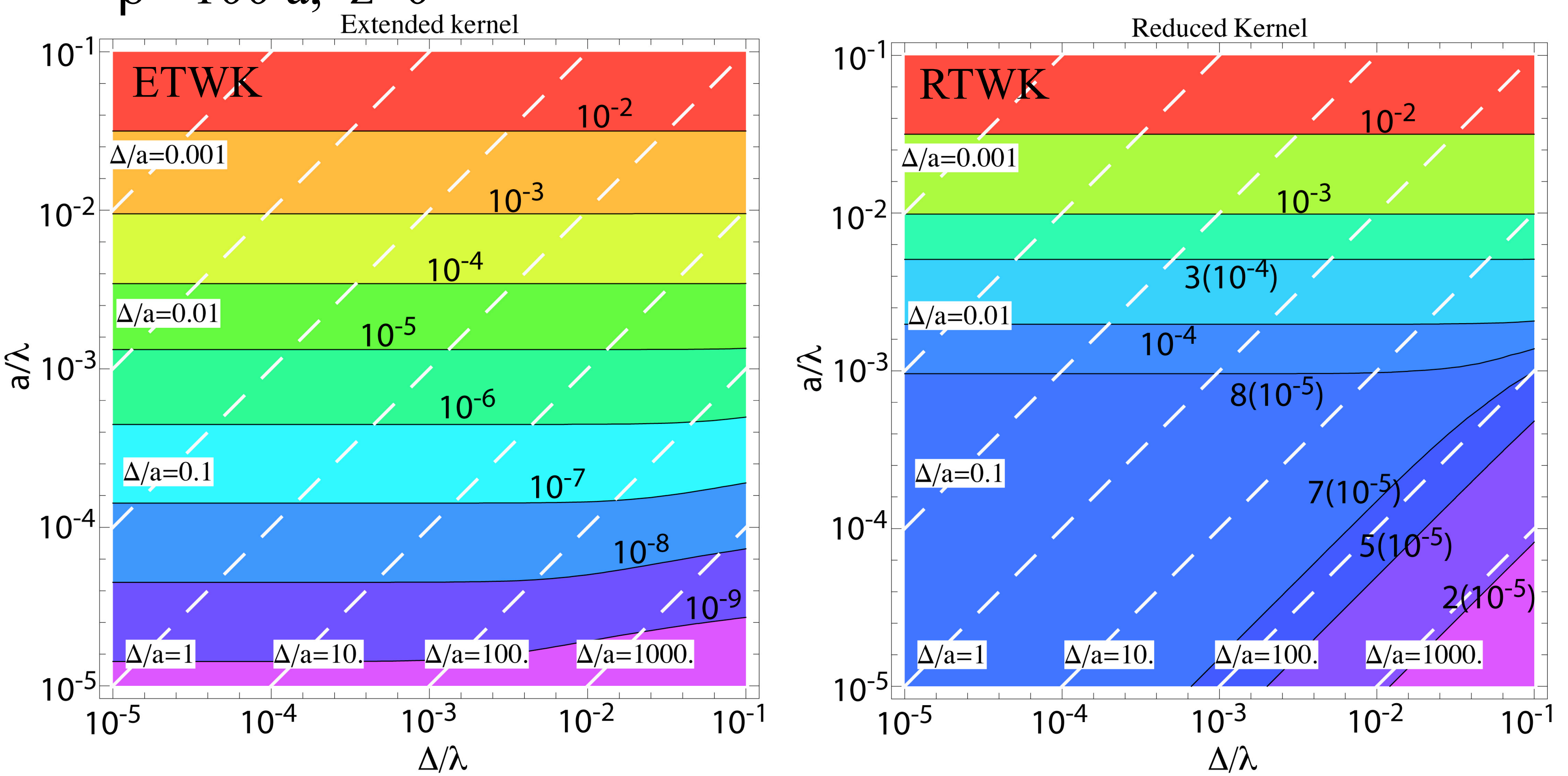
With increasing ρ the RTWK error gets smaller

$\rho = 10\ a,\ \ z=0$



With increasing ρ the RTWK error gets smaller

$\rho = 100\, a,\ \ z=0$



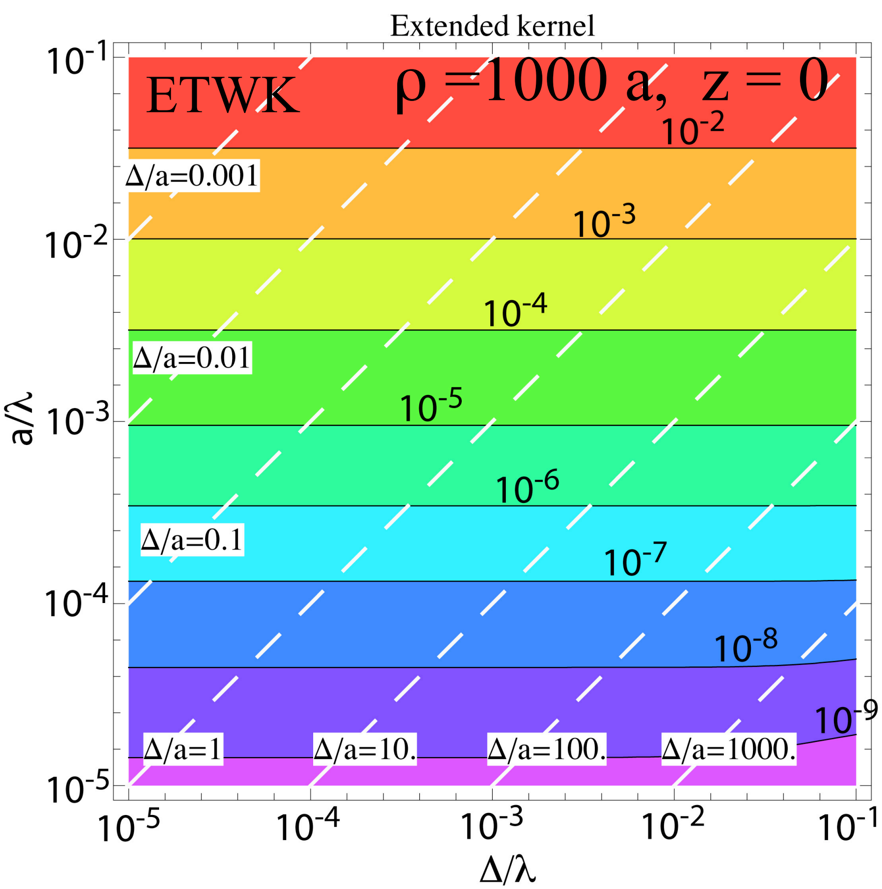
With ρ = 1000 a the RTWK is limited by phase error

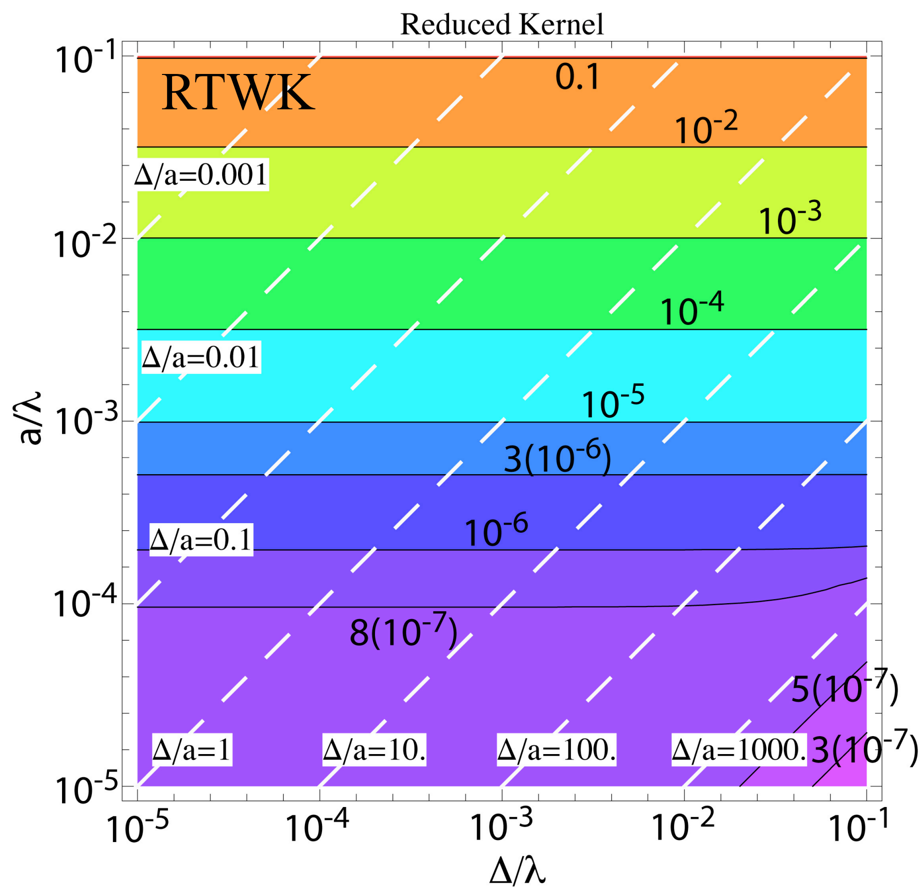
For $\rho >> a, \Delta$: $R \approx \rho - a\cos(\phi)$

Exact kernel ~ $\frac{1}{\rho}\int_{-\pi}^{\pi}\cos[ka\cos\phi]\,d\phi = 2\pi J_0(ka)/\rho$

ETWK, RTWK ~ $2\pi/\rho$

Error ~ $1 - J_0(ka)$

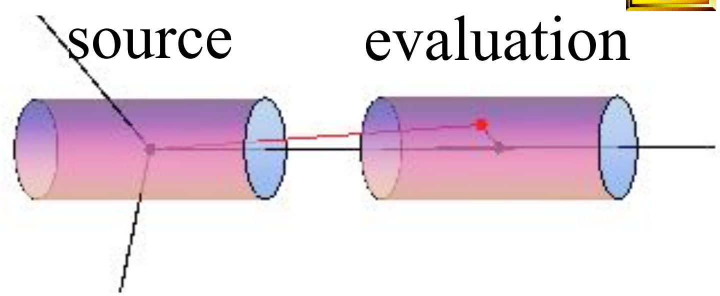




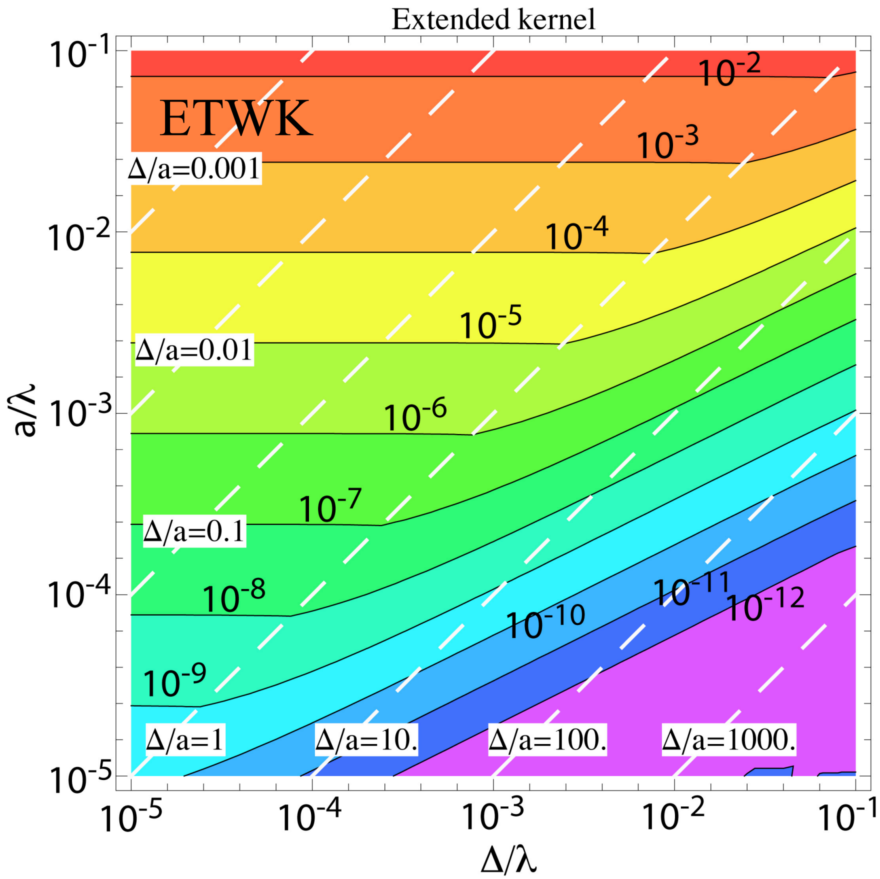
With increasing z the RTWK and ETWK errors get smaller

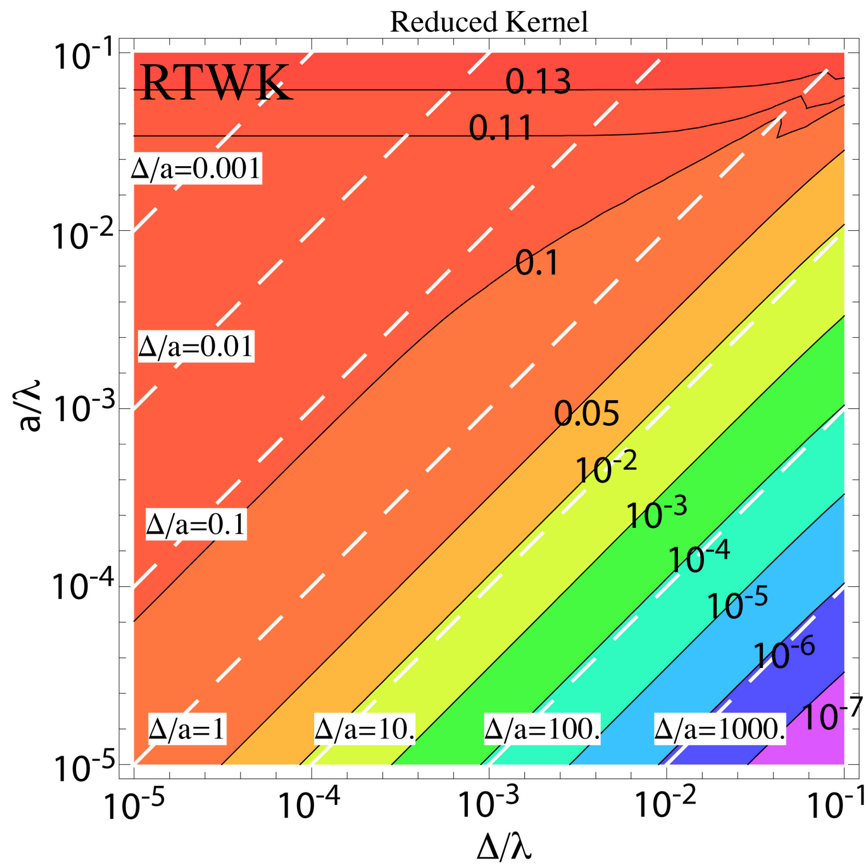
ETWK evaluation on surface

RTWK evaluation on axis



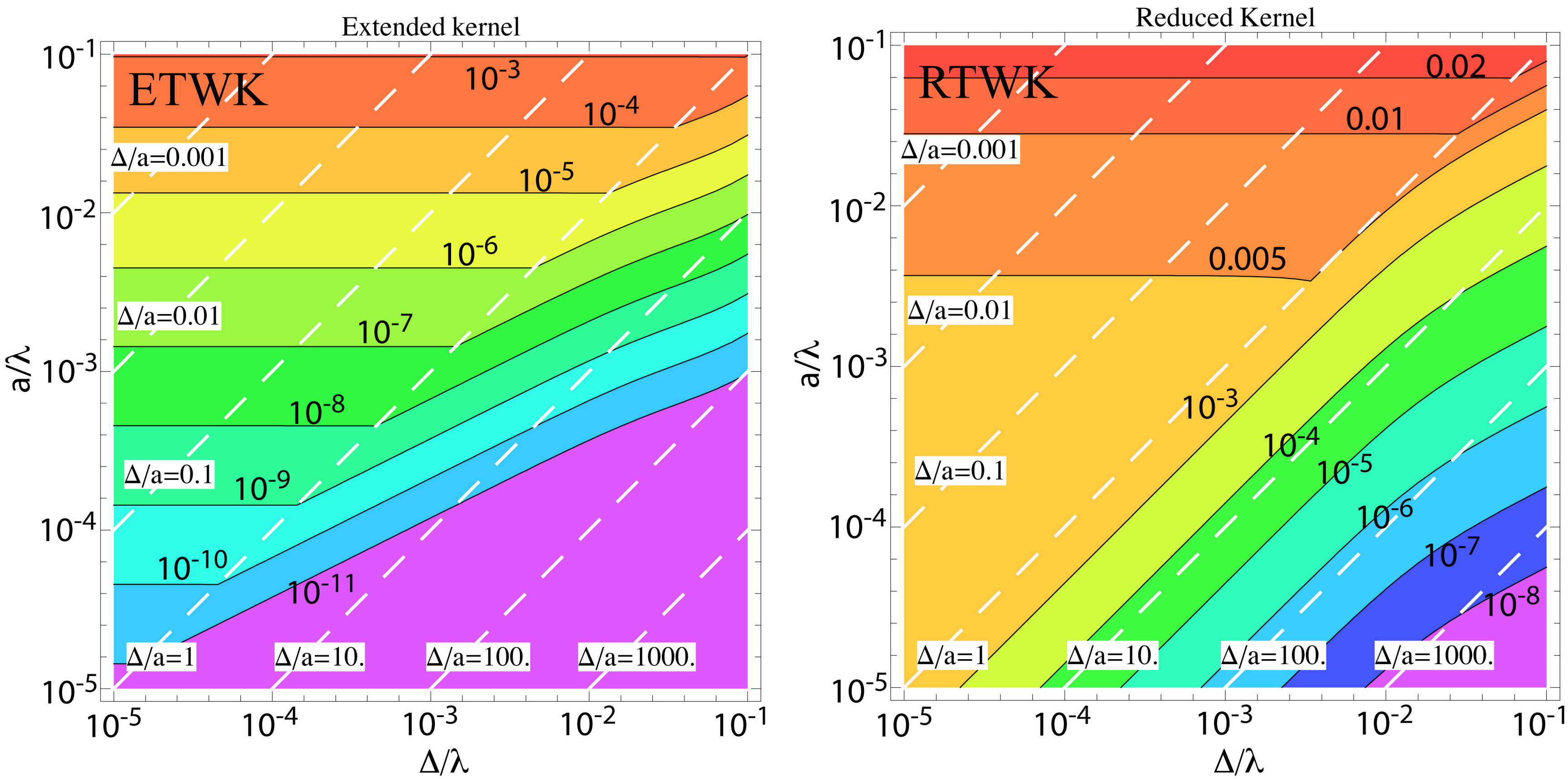
$z = 1 \max(\Delta, a), \rho = a$





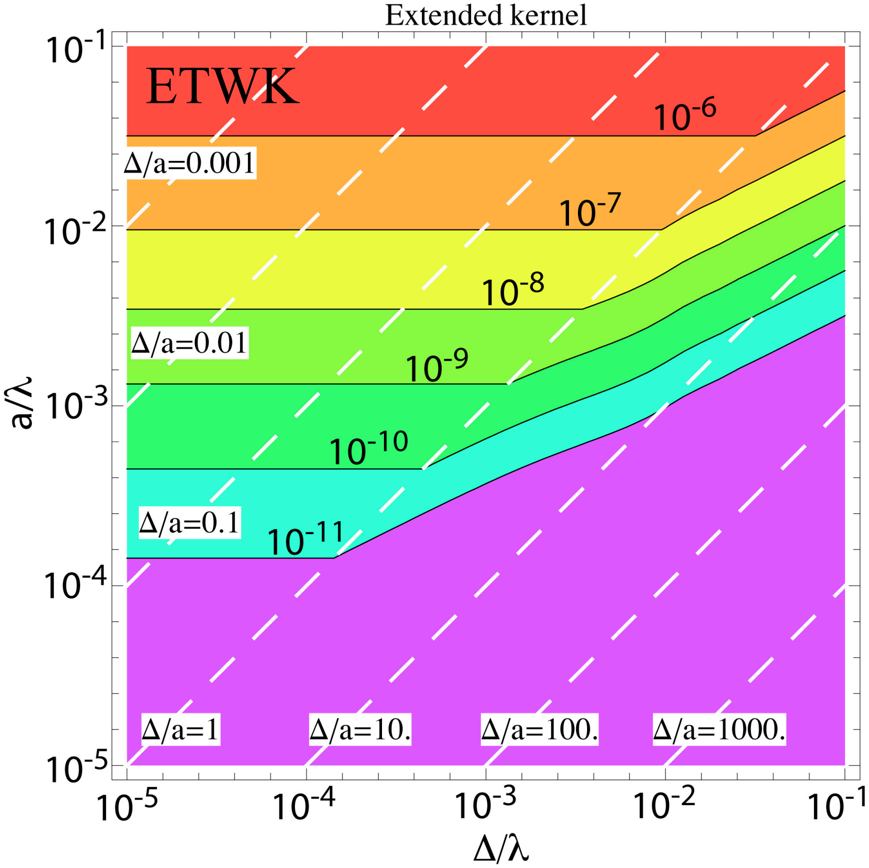
With increasing z the RTWK and ETWK errors get smaller

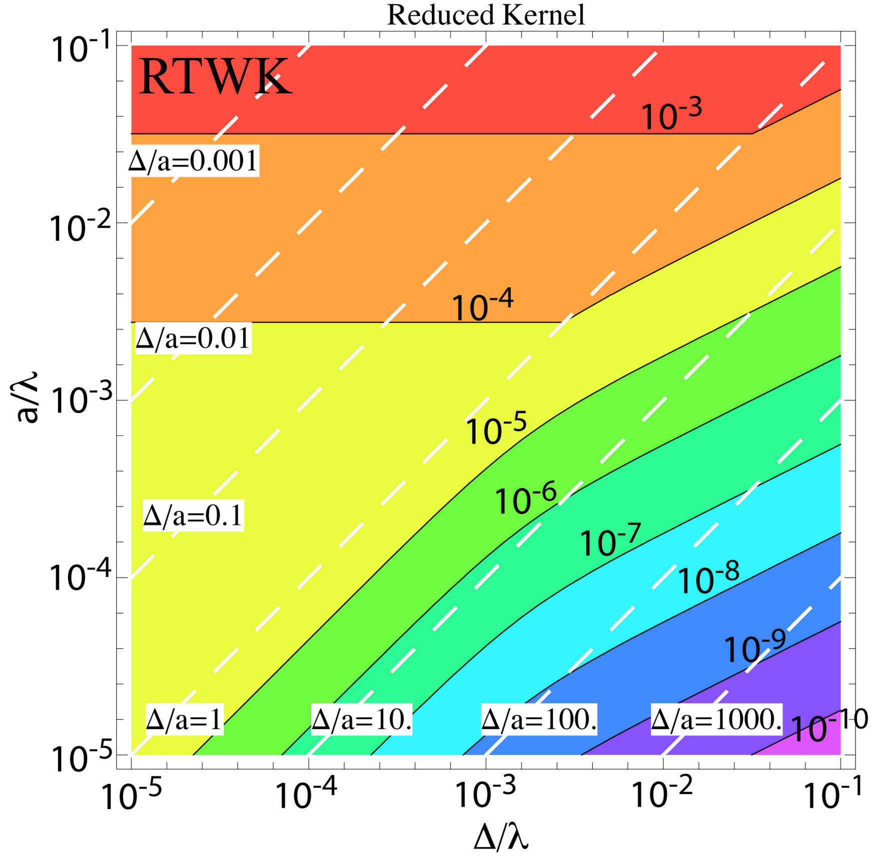
$z = 10 \max(\Delta, a)$, $\rho = a$



With increasing z the RTWK and ETWK errors get smaller

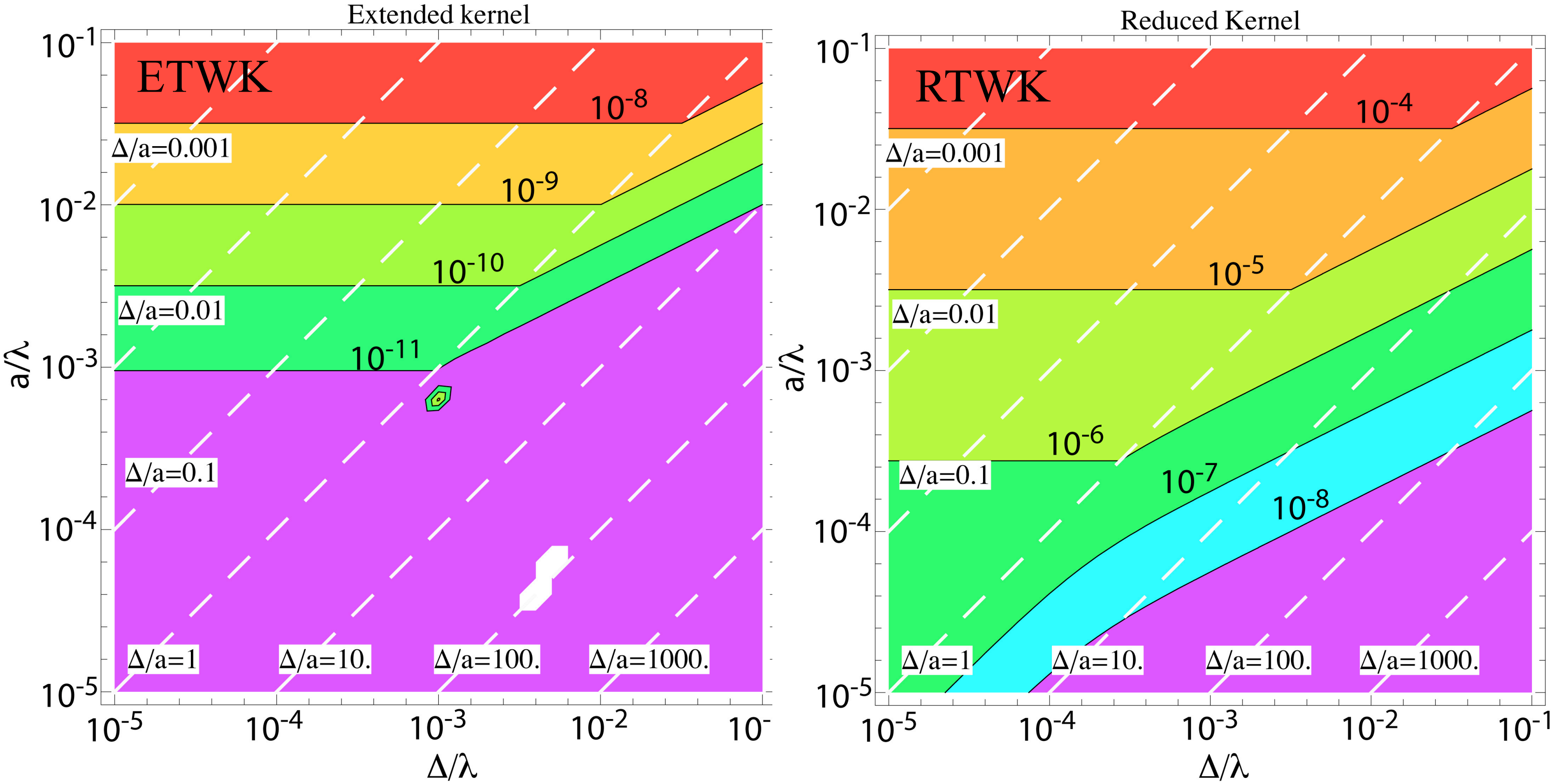
$z = 100 \max(\Delta, a)$, $\rho = a$





With increasing z the RTWK and ETWK errors get smaller

$z = 1000 \max(\Delta, a)$, $\rho = a$



NEC-4 and NEC-5 were compared with ETWK or RTWK

- NEC-5: mixed potential, linear basis, can use ETWK or RTWK

  In ETWK mode the RTWK is used to when R > 10 max(Δ,a)

- NEC-4: EFIE point-matched, RTWK only,

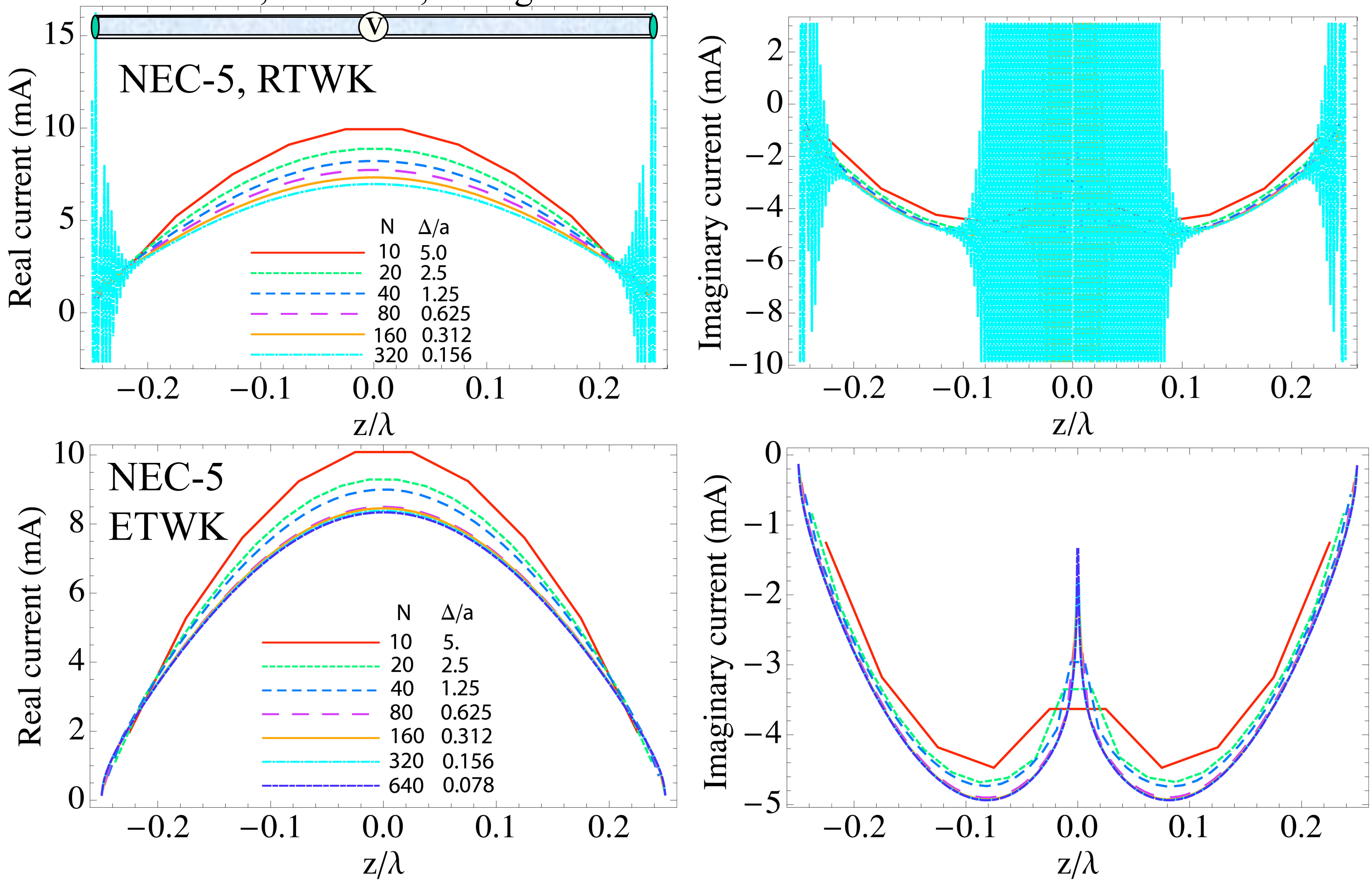
  Basis: A+ B sin(ks) + C cos(ks)

  Evaluation on the wire axis

  Approximate wire end caps

Dipole current is unstable with RTWK for small Δ/a but stable with ETWK

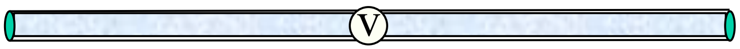
L = 0.5λ, a = 0.01λ, N segments

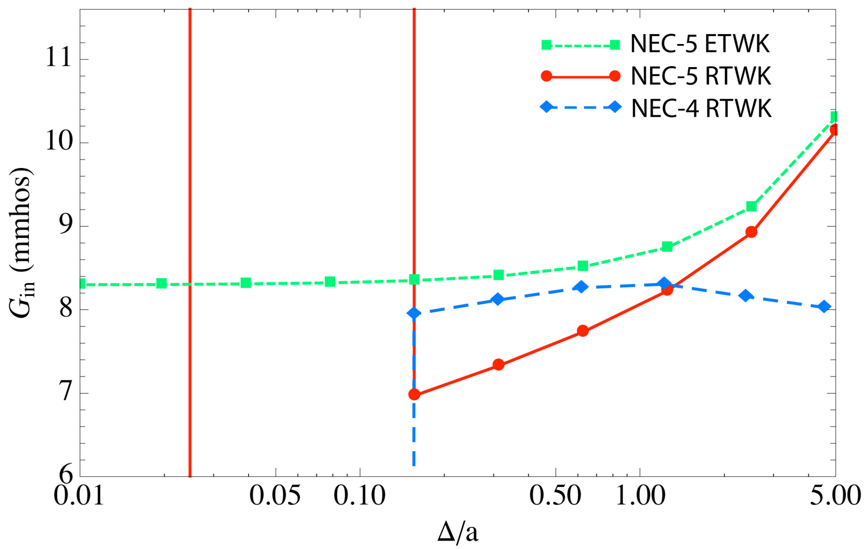


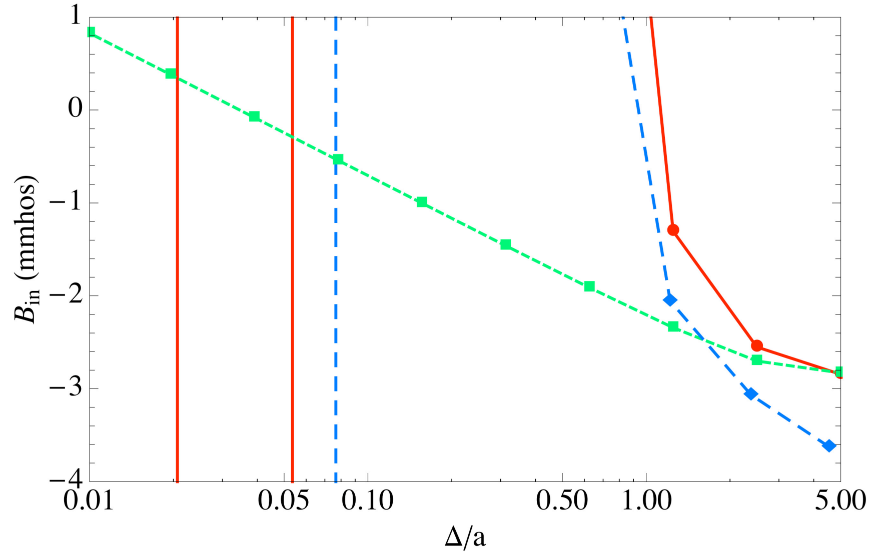
The dipole solution converges with the ETWK

NEC-4 and 5 solutions with RTWK blow up

L = 0.5λ, a = 0.01λ, N segments

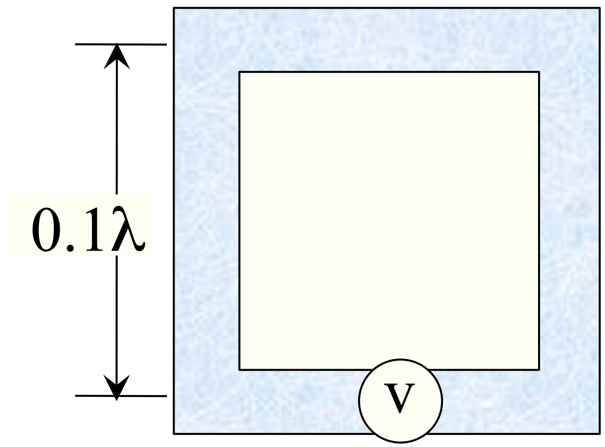


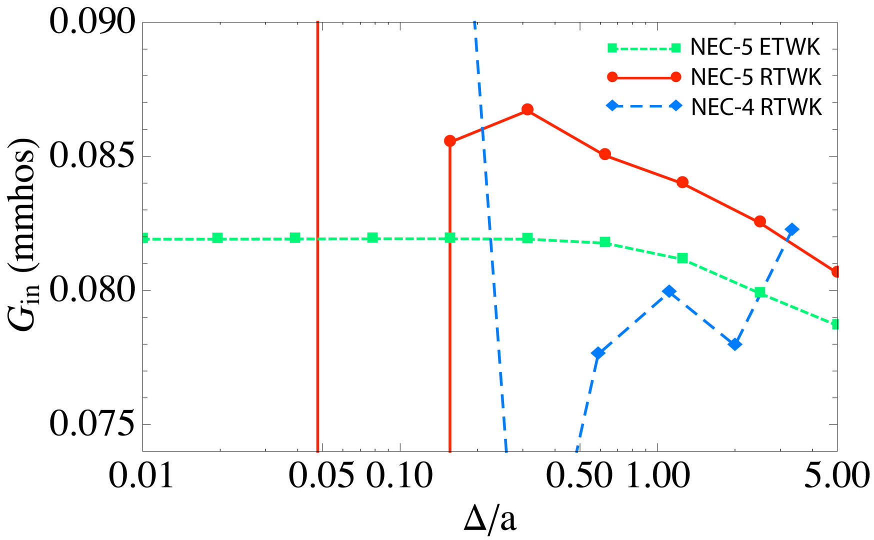


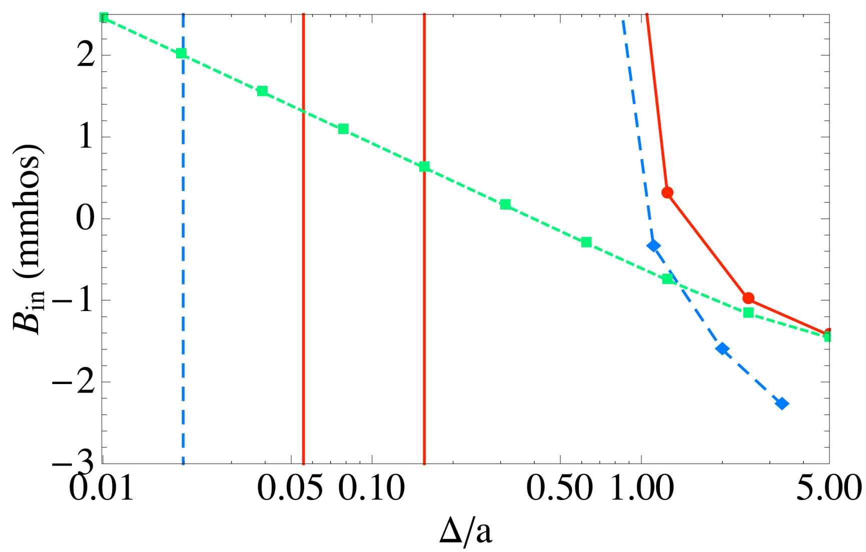


The ETWK converges for a square loop to small Δ/a

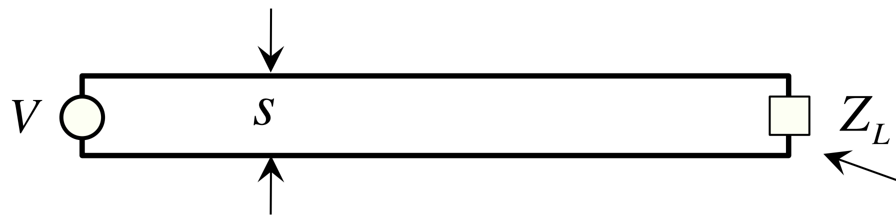
wire radius a = 0.01λ







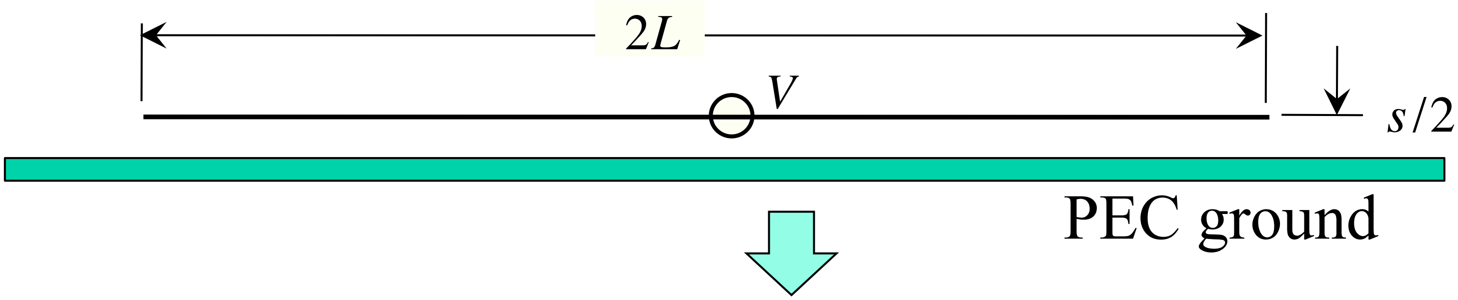
Another test of the kernel is the wire transmission line



wire diameter = d = 2a

small Δ/a

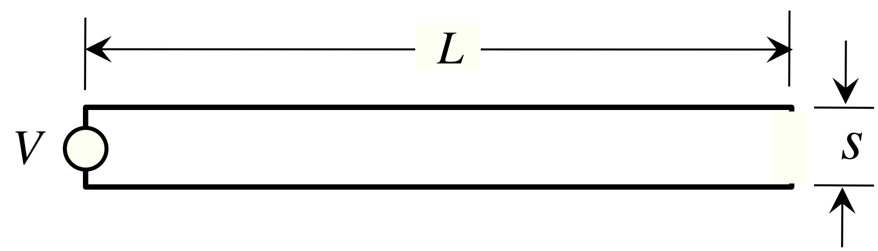
For an open termination model:



PEC ground

$$Z_{in} = -jZ_0 \cot(kL)$$

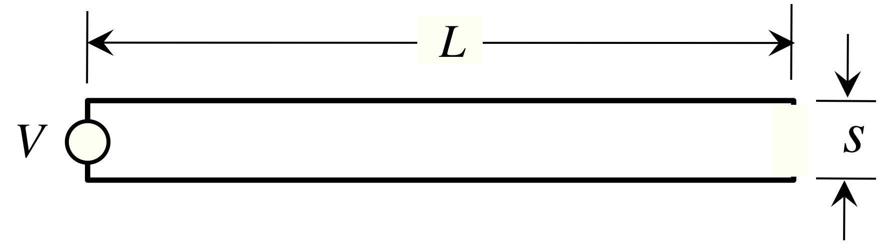
$$Z_0 = \frac{\eta_0}{\pi} \cosh^{-1}\left(\frac{s}{d}\right)$$



Transmission line equations were used to get Z_0 from Z_{in}

$$Y_{in} = \left(j / Z_0\right)\tan(kL)$$

$$Z_0 = \left(j / Y_{in}\right)\tan(kL)$$



Near kL = n(2π) use: $$\frac{dY_{in}}{df} = \frac{j2\pi L}{Z_0 c}\sec^2(kL)$$

$$Z_0 = \frac{j2\pi L \sec^2(kL)}{c\left(dY_{in} / df\right)}$$

computed

The model used:

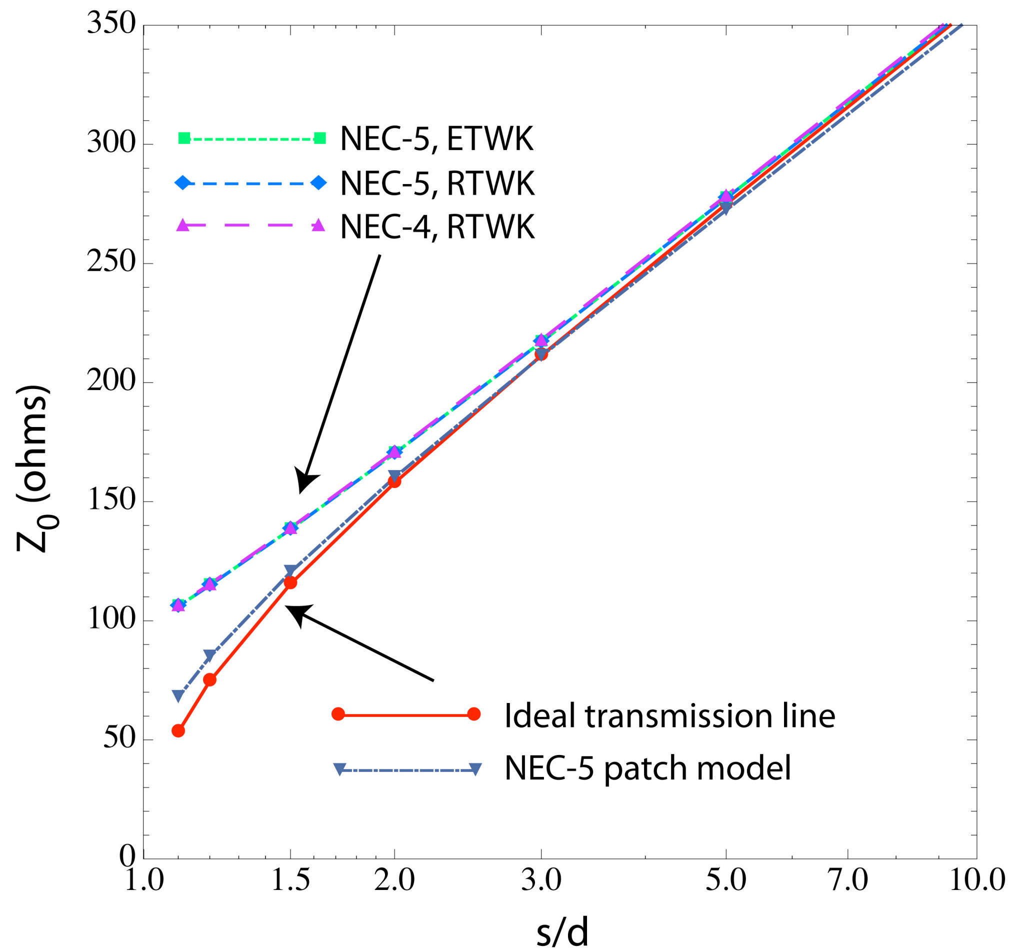
L = 5λ, a = 0.001λ

Number of segments in length 2L = 400

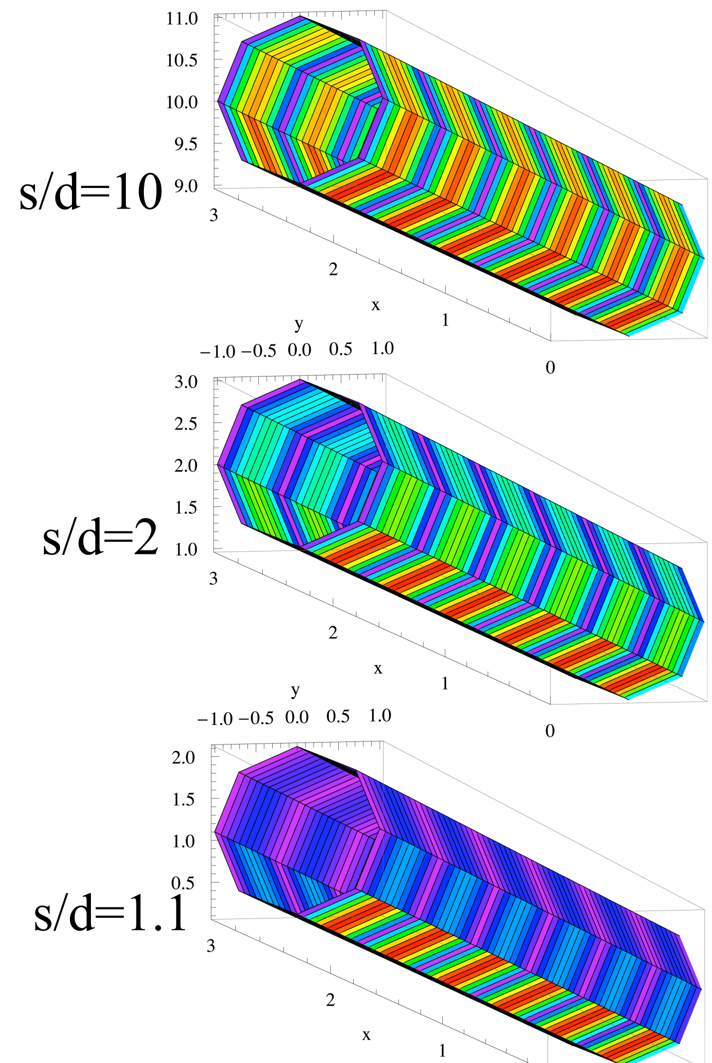
Δ/a= 25.

ETWK and RTWK make no difference on a transmission line

A patch model allows current to concentrate between the wires



Transverse dim. x1000

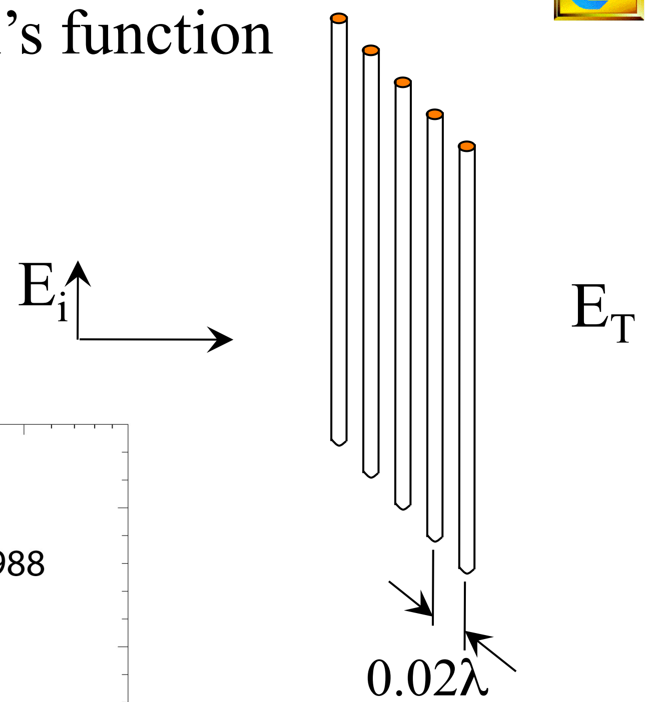


color: 3-decade log scale

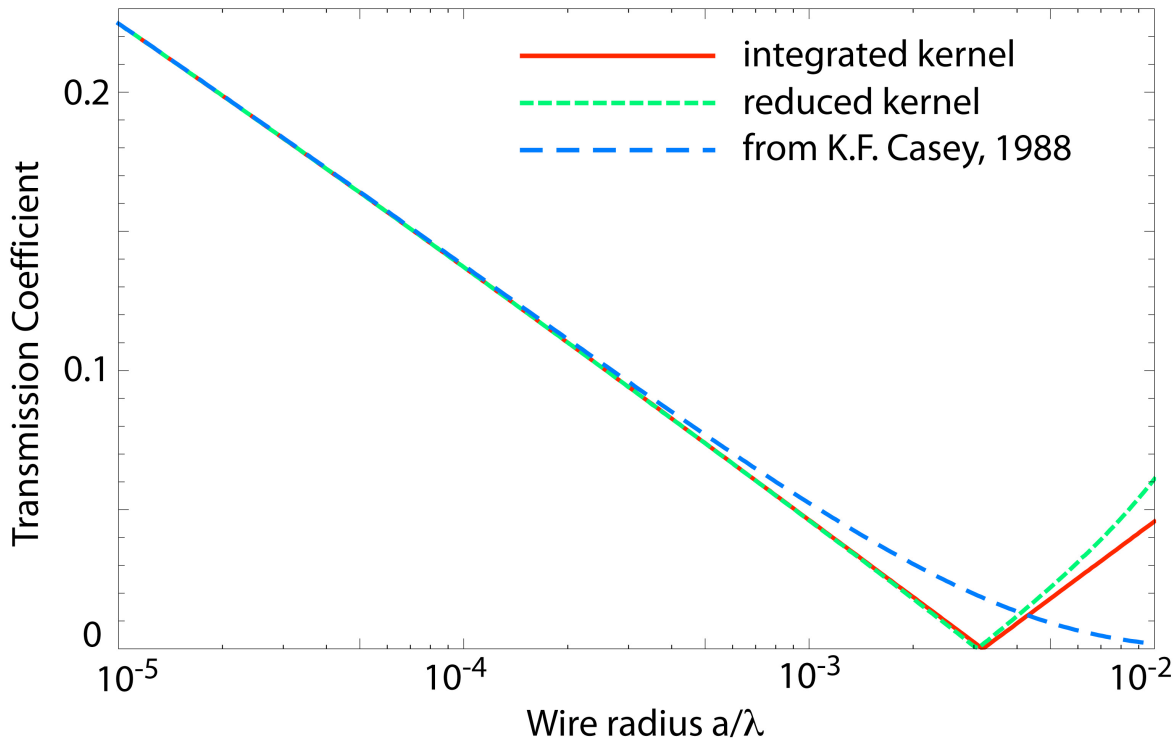
The "equal area rule" holds with reduced or integrated kernels

Infinite screen with 2D periodic Green's function

$H_0^{(2)}(k\rho)$ integrated numerically around the wire



Transmission coefficient = E_T/E_i



Conclusions:

- The ETWK allows stable convergence to small Δ/a

- Computational overhead for ETWK is minor

- It is still a thin-wire model, not accurate for large ka or wires close together or close to a ground plane

- “Equal area rule” for a wire mesh still applies with ETWK

- ETWK may give higher matrix condition number for large models (from Nathan Champagne)