LLNL-PROC-414517

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July 8, 2009

Recent Directions in Astrophysical Quantitative Spectroscopy (Dimitri-Fest)
Boulder, CO, United States
March 30, 2009 through April 3, 2009

Toward a Fully Consistent Radiation Hydrodynamics

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Abstract. Dimitri Mihalas set the standard for all work in radiation hydrodynamics since 1984. The present contribution builds on *Foundations of Radiation Hydrodynamics* to explore the relativistic effects that have prevented having a consistent non-relativistic theory. Much of what I have to say is in FRH, but the 3-D development is new. Results are presented for the relativistic radiation transport equation in the frame obtained by a Lorentz boost with the fluid velocity, and the exact momentum-integrated moment equations. The special-relativistic hydrodynamic equations are summarized, including the radiation contributions, and it is shown that exact conservation is obtained, and certain puzzles in the non-relativistic radhydro equations are explained. [1]

Keywords: radiation transport - relativistic, hydrodynamics - relativistic, radiation transport, comoving-frame
PACS: 95.30,Jx, 95.30,Lz, 95.30.Sf

1. INTRODUCTION

In an earlier presentation[1], I reviewed the computational methods for radiation hydrodynamic problems and found that there was no clear choice between the methods that solve for the radiation in the laboratory frame and those that solve for the radiation in the comoving frame of the fluid. In the words I used at that time,

> Several of the difficulties are associated with combining a somewhat relativistic treatment of radiation with a non-relativistic treatment of hydrodynamics. The principal problem is a tradeoff between easily obtaining the correct diffusion limit and describing free-streaming radiation with the correct wave speed. The computational problems of the comoving-frame formulation in more than one dimension, and the difficulty of obtaining both exact conservation and full $\mathbf{u}/c$ accuracy argue against this method. As the interest in multi-D increases, as well as the power of computers, the lab-frame method is becoming more attractive. The Monte Carlo method combines the advantages of both lab-frame and comoving-frame approaches, its only disadvantage being cost.

These remarks are true today, but I want to point out that there is a path to remove the trade-off between exact conservation and full $\mathbf{u}/c$ accuracy in the comoving-frame method, which is to use a fully-relativistic formulation.

[1] This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

The problem arises because the radiation field is described by an intensity function $I(\mathbf{r},\mathbf{n},\nu,t)$, and there is a choice of frame implicit in this function, either the laboratory frame — at rest with respect to the system as a whole, or the comoving frame — obtained from the former by a Lorentz transformation with the local fluid velocity. The direction vector $\mathbf{n}$ and the frequency ν change in the transformation, as does the value of I.[2] The transport operator is simple in the lab frame and the material-coupling terms are simple in the comoving frame, so both choices have good and bad aspects.

2. COMOVING TRANSPORT — THE NON-RELATIVISTIC VIEW

The exact transport equation in the lab frame is

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t}+\mathbf{n}\cdot\nabla I_\nu = j_\nu - k_\nu I_\nu \tag{1}$$

in which j_ν is the emissivity and k_ν is the absorptivity, and both include scattering. The effects of fluid motion are buried in j_ν and k_ν. The question arises whether or not to treat the kinematics relativistically in the radiation transport, when calculating the emission and absorption terms, or in the comoving-frame transport equation. The all-relativistic approach is consistent and recommended, but often we are called to couple radiation to non-relativistic hydrodynamics. In this case there will be inconsistencies if $O(u^2/c^2)$ terms are retained in the kinematics, which I want to explore next.

The relativistic Doppler and abberation transformations are the following

$$\nu = \nu_0\gamma_u\left(1+\frac{\mathbf{n}_0\cdot\mathbf{u}}{c}\right) \quad \text{and} \quad \mathbf{n} = \frac{\gamma_u\mathbf{u}/c+\mathbf{n}_0+(\gamma_u-1)(\mathbf{n}_0\cdot\mathbf{u})\mathbf{u}/u^2}{\gamma_u(1+\mathbf{n}_0\cdot\mathbf{u}/c)} \tag{2}$$

where "$_0$" quantities are in the comoving frame of the fluid, which moves with velocity $\mathbf{u}$, and $\gamma_u = (1-u^2/c^2)^{-1/2}$. In the non-relativistic case $u \ll c$ and the relations become

$$\nu = \nu_0\left(1+\frac{\mathbf{n}_0\cdot\mathbf{u}}{c}\right) \quad \text{and} \quad \mathbf{n} = \frac{\mathbf{n}_0+\mathbf{u}/c}{1+\mathbf{n}_0\cdot\mathbf{u}/c} \,. \tag{3}$$

Either relativistically or non-relativistically I_ν/ν^3 is a Lorentz invariant, so $I_\nu = (\nu/\nu_0)^3 I_\nu^0$.

The energy and momentum source terms that appear on the right-hand side of the non-relativistic Euler equations are the following:

$$g^0 = \int d\nu \int_{4\pi} d\Omega\,(j_\nu - k_\nu I_\nu) \quad \text{and} \quad \mathbf{g} = \frac{1}{c}\int d\nu \int_{4\pi} d\Omega\,\mathbf{n}(j_\nu - k_\nu I_\nu)\,. \tag{4}$$

The Euler equations themselves look like this:

$$\frac{\partial}{\partial t}(\rho e+\frac{1}{2}\rho u^2)+\nabla\boldsymbol{\cdot}(\rho\mathbf{u}h+\frac{1}{2}\rho\mathbf{u}u^2) = -g^0\,, \tag{5}$$

[2] At this meeting Ed Baron described the method of Chen, Kantowski, Baron, Knop and Hauschildt [3] that transforms ν but not $\mathbf{n}$; this has defi nite advantages.

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) + \nabla p = -\mathbf{g}\,. \tag{6}$$

Besides the usual symbols, h is the specific enthalpy of the material. Important note: the radiation terms are all in the lab frame here!

In the co-moving frame the energy and momentum coupling terms are evaluated in terms of the comoving-frame absorption and emission,

$$g_0^0 = \int d\nu_0 \int_{4\pi} d\Omega\,(j_\nu^0 - k_\nu^0 I_\nu^0) \quad \text{and} \quad \mathbf{g}_0 = \frac{1}{c}\int d\nu_0 \int_{4\pi} d\Omega\,\mathbf{n}_0 (j_\nu^0 - k_\nu^0 I_\nu^0)\,, \tag{7}$$

from which it follows, in the non-relativistic case, that

$$g^0 = g_0^0 + \mathbf{u}\cdot\mathbf{g}_0 \quad \text{and} \quad \mathbf{g} = \mathbf{g}_0 + \frac{\mathbf{u}}{c^2} g_0^0 \tag{8}$$

to order u/c. The second term in the equation for $\mathbf{g}$ is problematic. It is the same order as the momentum addition to the material caused by the increase of the relative mass density when the material gains energy, *i.e.,* a purely relativistic effect. I will return to this point after considering the fully-relativistic approach.

If we neglect the $\mathbf{u}/c^2$ term in the $\mathbf{g}$ equation, then the total material energy and momentum equations combine to yield the internal energy equation

$$\frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho\mathbf{u}e) + p\nabla\cdot\mathbf{u} = -g^0 + \mathbf{u}\cdot\mathbf{g} \approx -g_0^0\,. \tag{9}$$

Notice: the *internal* energy equation contains the radiation coupling in the comoving frame, while the *total* energy equation has the coupling term in the lab frame. We have to keep the frames straight! This also becomes clearer in the relativistic formulation below.

2.1. CMF Transport

The comoving-frame method describes the radiation using $\mathbf{n}_0$ and ν_0, the direction vector and frequency as viewed by an observer comoving with the fluid. This is a particular case of using an arbitrary tetrad $\{e_a^\mu, a = 1, \ldots, 4\}$ as the basis for 4-momentum space at each point $\{x^\mu\}$ of spacetime, where the e_a^μ are any desired functions. Thus the 4-momentum components in the natural basis and in the tetrad basis are related by

$$p^\mu = e_a^\mu p^a \tag{10}$$

The functions e_a^μ form a 4×4 matrix of which the inverse is the matrix e_μ^a. The crucial objects related to the e_a^μ are the Ricci rotation coefficients Ω_{bc}^a defined in the following way: Let a vector with tetrad components M^a and natural components $M^\alpha = e_a^\alpha M^a$ be displaced parallel to itself along $dx^\alpha = e_a^\alpha dx^a$. Parallel displacement requires that $dM^\alpha = -\Gamma_{\beta\gamma}^\alpha M^\beta dx^\gamma$, in terms of the Christoffel coefficients Γ of the basic manifold.

But the gradient in the tetrad functions also produces a change in the tetrad components for the displaced vector. The result is

$$
\begin{aligned}
&dM^a = -\Omega^a_{bc} M^b dx^c \\
&\text{with} \\
&\Omega^a_{bc} = e^a_\alpha e^\gamma_c e^\alpha_{b;\gamma} = e^a_\alpha e^\gamma_c e^\alpha_{b,\gamma} + e^a_\alpha e^\beta_b e^\gamma_c \Gamma^\alpha_{\beta\gamma}
\end{aligned} \tag{11}
$$

in which the comma and semicolon signify ordinary and covariant differentiation.

We let $\mathcal{I} \propto I_\nu/\nu^3$, $a \propto \nu k_\nu$ and $e \propto j_\nu/\nu^2$ denote the invariant intensity, absorptivity and emissivity, respectively. Let s be an affine parameter on the photon's null geodesic, so $dx^\mu/ds = p^\mu$, where p^μ is the 4-momentum. Then the invariant transport equation is

$$\frac{d\mathcal{I}}{ds} = e - a\mathcal{I} \tag{12}$$

The derivative on the left is evaluated using the result just found for dp^a, with $p^\mu = e^\mu_a p^a$ —

$$e^\mu_a p^a \mathcal{I}, \mu - \Omega^a_{bc} p^b p^c \frac{\partial \mathcal{I}}{\partial p^a} = e - a\mathcal{I} \ . \tag{13}$$

This is the general form of the invariant transport equation.

The $O(\mathbf{u}/c)$ equations for the moments of the radiation in the comoving frame can be found in references [6, 2, 1].

Summing the non-relativistic CMF energy equation and the product of $\mathbf{u}$ with the momentum equation, and conversely, *then discarding the higher-order terms in u*, leads to

$$\frac{\partial}{\partial t}\left(E_0 + \frac{2}{c^2}\mathbf{u}\cdot\mathbf{F}_0\right) + \nabla\cdot(\mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0) = g^0_0 + \mathbf{u}\cdot\mathbf{g}_0 \ . \tag{14}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\mathsf{P}_0) + \nabla\cdot\left[c\mathsf{P}_0 + \frac{1}{c}(\mathbf{u}\mathbf{F}_0 + \mathbf{F}_0\mathbf{u})\right] = c\mathbf{g}_0 + \frac{1}{c}\mathbf{u}g^0_0 \ , \tag{15}$$

which are equivalent to the lab-frame moment equations. Global energy and momentum conservation are obeyed only to $O(u/c)$ when the comoving-frame equations of that order are used.

3. THE RELATIVISTIC CMF FORMULATION

3.1. The Ricci rotation coefficients derived from the Lorentz basis

With the convention $x^0 = t, x^1 = x$, *etc.*, the choice for the tetrad given by the Lorentz-transformed natural basis is just the transformation matrix itself,

$$(e^\mu_a) = \begin{pmatrix} \gamma_u & \gamma_u \mathbf{u}^{\mathrm{T}}/c^2 \\ \gamma_u \mathbf{u} & \mathbf{I} + (\gamma_u - 1)\mathbf{u}\mathbf{u}^{\mathrm{T}}/u^2 \end{pmatrix} , \tag{16}$$

in which $\mathbf{u}$ is the fluid velocity (column-) vector, the superscript $^{\mathrm{T}}$ denotes the transpose, $\mathbf{I}$ is the 3×3 identity matrix, and γ_u is the Lorentz factor corresponding to $\mathbf{u}$,

$1/\sqrt{1-u^2/c^2}$. The inverse matrix (e^a_μ) is just the Lorentz transformation matrix with $\mathbf{u}$ replaced by $-\mathbf{u}$.

We need the derivatives of e_a^μ by $x^0 = t$ and x^i, $i = 1,\ldots,3$. Using equation (16) we find for the time derivative

$$\begin{aligned}(e^\mu_{a,0}) &= \gamma_u^3\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\begin{pmatrix}1 & \mathbf{u}^{\mathrm{T}}/c^2\\ \mathbf{u} & \mathbf{u}\mathbf{u}^{\mathrm{T}}/u^2\end{pmatrix}\\ &\quad+\begin{pmatrix}0 & \gamma_u\mathbf{a}^{\mathrm{T}}/c^2\\ \gamma_u\mathbf{a} & (\gamma_u-1)\left(\mathbf{a}\mathbf{u}^{\mathrm{T}}+\mathbf{u}\mathbf{a}^{\mathrm{T}}-2\mathbf{u}\cdot\mathbf{a}\mathbf{u}\mathbf{u}^{\mathrm{T}}/u^2\right)/u^2\end{pmatrix},\end{aligned} \tag{17}$$

with $\mathbf{a} = \partial\mathbf{u}/\partial t$, not to be confused with the fluid acceleration $\partial\mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u}$. Similarly the space derivatives become

$$\begin{aligned}(e^\mu_{a,i}) &= \gamma_u^3\frac{\mathbf{u}\cdot\partial_i\mathbf{u}}{c^2}\begin{pmatrix}1 & \mathbf{u}^{\mathrm{T}}/c^2\\ \mathbf{u} & \mathbf{u}\mathbf{u}^{\mathrm{T}}/u^2\end{pmatrix}\\ &\quad+\begin{pmatrix}0 & \gamma_u\partial_i\mathbf{u}^{\mathrm{T}}/c^2\\ \gamma_u\partial_i\mathbf{u} & (\gamma_u-1)\left[(\partial_i\mathbf{u})\mathbf{u}^{\mathrm{T}}+\mathbf{u}\partial_i\mathbf{u}^{\mathrm{T}}-2\mathbf{u}\cdot(\partial_i\mathbf{u})\mathbf{u}\mathbf{u}^{\mathrm{T}}/u^2\right]/u^2\end{pmatrix}.\end{aligned} \tag{18}$$

In order to get the Ricci rotation coefficients Ω^a_{bc} we have to combine these matrices with the transformation matrix and its inverse to form the object $e^a_\alpha e^\gamma_c e^\alpha_{b,\gamma}$. The results for (Ω^0_{bc}) and (Ω^i_{bc}) are found to be

$$(\Omega^0_{bc}) = \gamma_u\begin{pmatrix}0 & 0\\[2ex] \begin{matrix}\frac{\gamma_u}{c^2}(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})\\ +\frac{\gamma_u(\gamma_u-1)}{c^2u^2}\mathbf{u}\cdot(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})\mathbf{u}\end{matrix} & \begin{matrix}\gamma_u\mathbf{a}\mathbf{u}^{\mathrm{T}}/c^4+\gamma_u\frac{\gamma_u-1}{c^4u^2}\mathbf{u}\cdot\mathbf{a}\mathbf{u}\mathbf{u}^{\mathrm{T}}\\ +\nabla\mathbf{u}/c^2+\frac{\gamma_u-1}{c^2u^2}\mathbf{u}(\nabla\mathbf{u}\cdot\mathbf{u})\\ +\frac{\gamma_u-1}{c^2u^2}\mathbf{u}\cdot\nabla\mathbf{u}\mathbf{u}^{\mathrm{T}}+\frac{(\gamma_u-1)^2}{c^2u^4}[(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}]\mathbf{u}\mathbf{u}^{\mathrm{T}}\end{matrix}\end{pmatrix} \tag{19}$$

and

$$(\Omega^i_{bc}) = \begin{pmatrix}\begin{matrix}\gamma_u^2(a_i+\mathbf{u}\cdot\nabla u_i)\\ +\gamma_u^2(\gamma_u-1)\mathbf{u}\cdot(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})u_i/u^2\end{matrix} & \begin{matrix}\gamma_u^2a_i\mathbf{u}^{\mathrm{T}}/c^2+\gamma_u^2(\gamma_u-1)(\mathbf{u}\cdot\mathbf{a})u_i\mathbf{u}^{\mathrm{T}}/(c^2u^2)\\ +\gamma_u\nabla u_i+\gamma_u(\gamma_u-1)(\nabla\mathbf{u}\cdot\mathbf{u})u_i/u^2\\ +\gamma_u(\gamma_u-1)(\mathbf{u}\cdot\nabla u_i)\mathbf{u}^{\mathrm{T}}/u^2\\ +\gamma_u(\gamma_u-1)^2(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u})u_i\mathbf{u}^{\mathrm{T}}/u^4\end{matrix}\\[4ex] \begin{matrix}\frac{\gamma_u(\gamma_u-1)}{u^2}\big[-u_i(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})\\ +(a_i+\mathbf{u}\cdot\nabla u_i)\mathbf{u}\big]\end{matrix} & \begin{matrix}\frac{\gamma_u-1}{u^2}\Big\{-\gamma_uu_i\mathbf{a}\mathbf{u}^{\mathrm{T}}/c^2+\gamma_ua_i\mathbf{u}\mathbf{u}^{\mathrm{T}}/c^2\\ -u_i\nabla\mathbf{u}+\mathbf{u}\nabla u_i\\ +\frac{\gamma_u-1}{u^2}\big[-u_i(\mathbf{u}\cdot\nabla\mathbf{u})\mathbf{u}^{\mathrm{T}}+(\mathbf{u}\cdot\nabla u_i)\mathbf{u}\mathbf{u}^{\mathrm{T}}\big]\Big\}\end{matrix}\end{pmatrix}. \tag{20}$$

3.2. Comoving-frame transport equation

Equations (19) and (20) are the main results of this problem. We recall that the invariant transport equation is

$$e_a^\mu p^a \mathscr{I},\mu - \Omega^a_{bc} p^b p^c \frac{\partial \mathscr{I}}{\partial p^a} = \mathscr{e} - \mathscr{a}\mathscr{I} \ . \tag{21}$$

In working out the partial derivatives of $\mathscr{I}$ with respect to the momentum components, we note that $\mathscr{I}$ can be considered to be a function of three of them, since $\mathscr{I}$ is defined only on the null surface in momentum space. We choose the three space-like tetrad components, so the momentum derivative comes out in terms of Ω^i_{bc}. The derivative of ν_0 itself can be expressed using Ω^0_{bc}.

In terms of ordinary variables the transport equation becomes

$$\frac{dt}{ds}\frac{\partial I^0}{\partial t} + \frac{d\mathbf{r}}{ds}\cdot\nabla I^0 + \frac{d\mathbf{p}}{ds}\cdot\nabla_{\mathbf{p}} I^0 - \frac{3}{\nu_0}\frac{d\nu_0}{ds} I^0 = j^0 - k^0 I^0 \ . \tag{22}$$

The coefficients dt/ds and $d\mathbf{r}/ds$ are the time and space components of $p^\mu = e_a^\mu p^a$, divided by $\nu_0 = \mathscr{a}/k^0$ for convenience, so they are

$$\frac{dt}{ds} = \gamma_u (1 + \mathbf{u}\cdot\mathbf{n}_0/c)/c \quad \text{and} \quad \frac{d\mathbf{r}}{ds} = \gamma_u \mathbf{u}/c + \mathbf{n}_0 + (\gamma_u - 1)\mathbf{u}\cdot\mathbf{n}_0\mathbf{u}/u^2 \ . \tag{23}$$

The coefficients $d\mathbf{p}/ds$ are $-(1/\nu_0)\Omega^i_{bc} p^b p^c$. These are given by

$$\begin{aligned}
\frac{d\mathbf{p}}{ds} = -\frac{\nu_0}{c}\Bigg\{ & \frac{1}{c}\gamma_u^2(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u}) + \frac{\gamma_u^2(\gamma_u-1)}{cu^2}\mathbf{u}\cdot(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})\,\mathbf{u} \\
& + \frac{\gamma_u(\gamma_u-1)}{u^2}\big[-(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{n}_0\,\mathbf{u} + (\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})\mathbf{u}\cdot\mathbf{n}_0\big] \\
& + \frac{\gamma_u^2}{c^2}\mathbf{a}\,(\mathbf{u}\cdot\mathbf{n}_0) + \frac{\gamma_u^2(\gamma_u-1)}{c^2u^2}(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{n}_0)\,\mathbf{u} + \gamma_u\mathbf{n}_0\cdot\nabla\mathbf{u} \\
& + \frac{\gamma_u(\gamma_u-1)}{u^2}\big[(\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{u})\,\mathbf{u} + (\mathbf{u}\cdot\nabla\mathbf{u})\,(\mathbf{u}\cdot\mathbf{n}_0)\big] \\
& + \frac{\gamma_u(\gamma_u-1)^2}{u^4}(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u})\,(\mathbf{u}\cdot\mathbf{n}_0)\,\mathbf{u} \\
& + \frac{\gamma_u-1}{u^2}\Big\{-\gamma_u\mathbf{u}\,(\mathbf{n}_0\cdot\mathbf{a})\,(\mathbf{u}\cdot\mathbf{n}_0)/c + \gamma_u\mathbf{a}\,(\mathbf{u}\cdot\mathbf{n}_0)^2/c \\
& - c\mathbf{u}\,(\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0) + c\mathbf{n}_0\cdot\nabla\mathbf{u}\,(\mathbf{u}\cdot\mathbf{n}_0) \\
& + \frac{\gamma_u-1}{u^2}\big[-c\mathbf{u}\,(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0)\,(\mathbf{u}\cdot\mathbf{n}_0) + c(\mathbf{u}\cdot\nabla\mathbf{u})\,(\mathbf{u}\cdot\mathbf{n}_0)^2\big]\Big\}\Bigg\} \ .
\end{aligned} \tag{24}$$

The value of $d\nu_0/ds$ is $-(1/\nu_0)\Omega^0_{bc}p^b p^c$, and in view of equation (19) this is

$$
\begin{aligned}
\frac{d\nu_0}{ds} = -\frac{\nu_0}{c}\Bigg\{ & \frac{\gamma_u^2}{c}(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{n}_0 + \frac{\gamma_u^2(\gamma_u-1)}{cu^2}\mathbf{u}\cdot(\mathbf{a}+\mathbf{u}\cdot\nabla\mathbf{u})(\mathbf{u}\cdot\mathbf{n}_0) \\
& + \frac{\gamma_u^2}{c^2}(\mathbf{n}_0\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{n}_0) + \frac{\gamma_u^2(\gamma_u-1)}{c^2u^2}(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}\cdot\mathbf{n}_0)^2 + \gamma_u\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0 \\
& + \frac{\gamma_u(\gamma_u-1)}{u^2}(\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{n}_0) + \frac{\gamma_u(\gamma_u-1)}{u^2}(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0)(\mathbf{u}\cdot\mathbf{n}_0) \\
& + \frac{\gamma_u(\gamma_u-1)^2}{u^4}(\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{n}_0)^2 \Bigg\} \ .
\end{aligned}
\tag{25}
$$

A consistency condition is that

$$
\mathbf{n}_0\cdot\frac{d\mathbf{p}}{ds} \equiv \frac{d\nu_0}{ds} \ , \tag{26}
$$

which ensures that the null vector p^μ remains null during transport. As we see, equations (24) and (25) guarantee this.

During this workshop I learned from Ed Baron about the work of Morita and Kaneko [7], who provide a relativistic CMF formulation comparable to the present one. I have not so far verified if it is equivalent since it uses spherical polar coordinates in **p**-space. The method of Chen, *et al.*, [3] is also quite interesting. Rather than use *any* Lorentz transformation as a basis in momentum space, these authors adopt lab-frame angles and comoving-frame frequency as the independent variables, and derive the transport equation using the chain rule. The negative aspect of this is that obtaining the components of the stress tensor may be awkward.

3.3. Conservative form of the transport equation

A simple manipulation of the transport equation (21) puts it into the form of a conservation law:

$$
(e^\mu_a p^a \mathcal{I})_{;\mu} - \frac{\partial}{\partial p^a}\left(\Omega^a_{bc}p^b p^c \mathcal{I}\right) = e - a\,\mathcal{I} \ . \tag{27}
$$

It is the covariant derivative that appears in the first term on the left, and in this form the equation is valid in curvilinear coordinates and in general relativity. We use the fact that covariant differentiation obeys the product rule [4]. In these cases the Christoffel coefficients $\Gamma^\alpha_{\beta\gamma}$ based on the appropriate metric tensor must be included. In order to prove that equations (21) and (27) are equivalent, we have to demonstrate the following identity:

$$
e^\mu_{a;\mu}p^a \equiv \frac{\partial}{\partial p^a}\left(\Omega^a_{bc}p^b p^c\right) = \frac{\partial}{\partial p^a}\left(e^a_\alpha e^\gamma_c e^\alpha_{b;\gamma}p^b p^c\right) \tag{28}
$$

independently of the p^a. In fact, the expansion of the right-hand side gives

$$
e^a_\alpha e^\gamma_c e^\alpha_{a;\gamma}p^c + e^a_\alpha e^\gamma_a e^\alpha_{b;\gamma}p^b \ . \tag{29}
$$

It is not hard to show that $e^a_\alpha e^\alpha_{a;\gamma} = 0$.[3] That leaves the second term in the expression. But $e^a_\alpha e^\gamma_a = \delta^\gamma_\alpha$, so the second term becomes $e^\alpha_{b;\alpha} p^b$ and the identity is proved.

There is a hidden subtlety in equation (27). It is that the momentum-space divergence is written in four-dimensional form, while we regard $\mathscr{I}$ as a function only of the 3-space components $\mathbf{p}$, and it is defined only on the null cone. An integral over a 4-volume including a patch of the null cone collapses to a surface integral with the invariant momentum volume element $d^3\mathbf{p}/p^0$.

3.4. Integrals of the transport equation

If equation (27) is integrated over momentum 4-space, the momentum derivative vanishes and the integrals of the other terms collapse to the integral on the null cone with $d^3\mathbf{p}/p^0$, as mentioned above. The first term becomes the 4-divergence of the number-flux vector N^μ, of which the fluid-frame components are

$$(N^a) = \int \frac{d^3\mathbf{p}_0}{\nu_0} \left(\nu_0 \quad c\nu_o \mathbf{n}_0\right) \mathscr{I} = c^3 \int \nu_0 d\nu_0 d\Omega \left(\nu_0 \quad c\nu_0 \mathbf{n}_0\right) \mathscr{I} \; . \tag{30}$$

In terms of the ordinary intensity this is

$$(N^a) = \int \frac{d\nu_0 d\Omega}{h\nu_0} \left(1 \quad c\mathbf{n}_0\right) I^0 \; . \tag{31}$$

The factors of c^3 have been absorbed into the definition of $\mathscr{I}$, *viz.*, $I^0 = c^3 \nu_0^3 \mathscr{I}$, and similarly for $\mathscr{e}$, and a factor h was included for dimensional consistency. The terms on the right-hand side integrate to

$$\int \frac{d^3\mathbf{p}_0}{h\nu_0} (\mathscr{e} - \mathscr{a}\mathscr{I}) = \frac{c^3}{h} \int \nu_0 d\nu_0 d\Omega \, (\mathscr{e} - \mathscr{a}\mathscr{I}) = \int \frac{d\nu_0 d\Omega}{h\nu_0} (j^0_{\nu_0} - k^0_{\nu_0} I^0) \; , \tag{32}$$

so the photon number conservation law is

$$(e^\mu_a N^a)_{;\mu} = \int \frac{d\nu_0 d\Omega}{h\nu_0} (j^0_{\nu_0} - k^0_{\nu_0} I^0) \; , \tag{33}$$

with N^a given by equation (31).

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[3] $e^a_\alpha e^\alpha_{a;\gamma} = e^a_\alpha e^\alpha_{a,\gamma} + e^a_\alpha \Gamma^\alpha_{\beta\gamma} e^\beta_a$, and the second term becomes $\Gamma^\alpha_{\alpha\gamma}$, which equals $(1/2)\ln(g)_{,\gamma}$ [4]. If A is any matrix, and dA is its differential, Jacobi's rule for differentiating a determinant is that $d\ln\det(A) = \mathrm{Tr}(A^{-1}dA)$. When applied to $A = (e^\alpha_a)$, with $\det(e) \propto 1/\sqrt{g}$, we get $e^a_\alpha e^\alpha_{a,\gamma} = (\ln\det(e))_{,\gamma} = -(1/2)\ln(g)_{,\gamma}$, canceling the second term.

Next we multiply equation (27) by the lab-frame momentum $p^\nu = e^\nu_d p^d$. The left side becomes

$$e^\nu_d p^d \left[(e^\mu_a p^a \mathcal{I})_{;\mu} - \frac{\partial}{\partial p^a} \left(e^a_\alpha e^\gamma_c e^\alpha_{b;\gamma} p^b p^c \mathcal{I} \right) \right] = (e^\mu_a e^\nu_d p^a p^d \mathcal{I})_{;\mu} - \frac{\partial}{\partial p^a} \left(e^a_\alpha e^\gamma_c e^\nu_d e^\alpha_{b;\gamma} p^b p^c p^d \mathcal{I} \right) - e^\mu_a e^\nu_{d;\mu} p^a p^d \mathcal{I} + e^a_\alpha e^\gamma_c e^\nu_d e^\alpha_{b;\gamma} p^b p^c \mathcal{I} \delta^d_a \ . \tag{34}$$

The Kronecker-δ in the last term makes that term into $e^d_\alpha e^\gamma_c e^\nu_d e^\alpha_{b;\gamma} p^b p^c \mathcal{I}$; then the product $e^d_\alpha e^\nu_d$ sums to δ^ν_α, since it is the product of a matrix with its inverse. Finally the last term becomes $e^\gamma_c e^\nu_{b;\gamma} p^b p^c \mathcal{I}$, which just cancels the second-last term. The equation we are left with is

$$(e^\mu_a e^\nu_d p^a p^d \mathcal{I})_{;\mu} - \frac{\partial}{\partial p^a} \left(e^a_\alpha e^\gamma_c e^\nu_d e^\alpha_{b;\gamma} p^b p^c p^d \mathcal{I} \right) = e^\nu_d p^d (\mathit{e} - \mathit{a}\mathcal{I}) \ . \tag{35}$$

The integral of this over 4-momentum space, divided by c^2, then leads to

$$T^{\nu\mu}_{;\mu} = c e^\nu_d \int \frac{d^3\mathbf{p}_0}{\nu_0} p^d (\mathit{e} - \mathit{a}\mathcal{I}) = e^\nu_d \frac{1}{c^2} \int d\nu_0 d\Omega \left(1 \quad c\mathbf{n}_0\right)^d (j^0_{\nu_0} - k^0_{\nu_0} I^0) \ . \tag{36}$$

in which $T^{\nu\mu}$ represents the lab-frame components of the energy-momentum tensor,

$$T^{\nu\mu} = c e^\mu_a e^\nu_d \int \frac{d^3\mathbf{p}_0}{\nu_0} p^a p^d \mathcal{I} = e^\mu_a e^\nu_d \frac{1}{c^2} \int d\nu_0 d\Omega \left(1 \quad c\mathbf{n}_0\right)^a \left(1 \quad c\mathbf{n}_0\right)^d I^0 \ . \tag{37}$$

The integrals in equations (36) and (37) are the comoving-frame source 4-vector and stress-energy tensor, respectively, and the e factors transform them into the lab frame.

The upshot of this section is that the relativistic comoving-frame transport equation is in precise agreement with the conservation law for the radiation stress-energy tensor that is in common use.

4. RELATIVISTIC HYDRODYNAMICS

The SRHD topic has been very well treated over the years. I can refer to Gerry Pomraning's book [5] and of course to Chapter 4 of Mihalas and Mihalas [6]. A very useful recent (and current) review is the on-line page by Martí and Müller [8], which has a complete basic exposition and includes a survey of the recent work.

The two basic equation of SRHD are continuity or baryon conservation

$$(\rho U^\mu)_{;\mu} = 0 \quad \text{and energy-momentum conservation} \quad ({}^{\mathrm{M}}T^{\mu\nu})_{;\nu} = f^\mu \tag{38}$$

in covariant form. The new quantities here are ρ, the baryon density in the fluid frame converted to a mass density with a common mass m_0 per baryon; $(U^\mu) = \gamma_u \left(1 \quad \mathbf{u}\right)$, the fluid 4-velocity; $({}^{\mathrm{M}}T^{\mu\nu})$, the material stress-energy tensor, and f^{μ}, the volume 4-force acting on the material. The covariant expression for the stress-energy tensor is

$$ {}^{\mathrm{M}}T^{\mu\nu} = \left(\rho + \frac{\rho e}{c^2} + \frac{p}{c^2}\right) U^{\mu} U^{\nu} - \frac{p}{c^2} g^{\mu\nu} \; . \tag{39} $$

The $g^{\mu\nu}$ that appear here are the contravariant components of the metric tensor; in the cartesian coordinates of flat space-time they comprise the Minkowski tensor $\mathrm{diag}(1, -c^2, -c^2, -c^2)$. The energy-momentum tensor space and time components in the lab frame become

$$ ({}^{\mathrm{M}}T^{\mu\nu}) = \begin{pmatrix} \rho\left(1+\frac{e}{c^2}\right)\gamma_u^2 + \frac{p}{c^2}(\gamma_u^2 - 1) & \left(\rho + \frac{\rho e}{c^2} + \frac{p}{c^2}\right)\gamma_u^2 \mathbf{u}^{\mathrm{T}} \\ \left(\rho + \frac{\rho e}{c^2} + \frac{p}{c^2}\right)\gamma_u^2 \mathbf{u} & p\mathbf{I} + \left(\rho + \frac{\rho e}{c^2} + \frac{p}{c^2}\right)\gamma_u^2 \mathbf{u}\mathbf{u}^{\mathrm{T}} \end{pmatrix} . \tag{40} $$

The conservation equation can be written out in terms of these components. The continuity equation becomes

$$ \frac{\partial}{\partial t}(\rho\gamma_u) + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\gamma_u\mathbf{u}) = 0 \; , \tag{41} $$

exactly the NR continuity equation except that ρ is multiplied by the Lorentz factor. The $\mu = 0$ component of the stress-energy conservation equation is

$$ \frac{\partial}{\partial t}\left[\rho\left(1+\frac{e}{c^2}\right)\gamma_u^2 + \frac{p}{c^2}(\gamma_u^2 - 1)\right] + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\rho + \frac{\rho e}{c^2} + \frac{p}{c^2}\right)\gamma_u^2\mathbf{u}\right] = f^0 \; . \tag{42} $$

By subtracting equation (41) from equation (42), then multiplying the result by c^2, the energy equation results in this form:

$$ \begin{aligned} &\frac{\partial}{\partial t}\left[\rho c^2\gamma_u(\gamma_u - 1) + \rho e\gamma_u^2 + p(\gamma_u^2 - 1)\right] \\ &\qquad + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\rho c^2\gamma_u(\gamma_u - 1)\mathbf{u} + \rho e\gamma_u^2\mathbf{u} + p\gamma_u^2\mathbf{u}\right] = c^2 f^0 \; . \end{aligned} \tag{43} $$

This is the conservation equation for total material energy. In the NR limit, the first term inside the time derivative tends to the kinetic energy density and the second to the internal energy density, while the pressure term becomes negligible. In the flux term, the first term in the flux is the kinetic energy flux, and the second and third comprise the enthalpy flux.

The $\mu = i$ components of the stress-energy conservation equation give the momentum equation,

$$ \frac{\partial}{\partial t}\left[\left(\rho + \frac{\rho e}{c^2} + \frac{p}{c^2}\right)\gamma_u^2\mathbf{u}\right] + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[p\mathbf{I} + \left(\rho + \frac{\rho e}{c^2} + \frac{p}{c^2}\right)\gamma_u^2\mathbf{u}\mathbf{u}\right] = \mathbf{f} \; . \tag{44} $$

By subtracting $(1 + e/c^2 + p/(\rho c^2))\gamma_u\mathbf{u}$ times the continuity equation (41) from this equation we get a form of acceleration equation,

$$ \rho\gamma_u\frac{D}{Dt}\left[\left(1 + \frac{e}{c^2} + \frac{p}{\rho c^2}\right)\gamma_u\mathbf{u}\right] = -\boldsymbol{\nabla}p + \mathbf{f} \; . \tag{45} $$

As Mihalas and Mihalas [6] show, following Thomas [9], the contraction of the 4-velocity vector times the stress-energy conservation law (38) reduces to

$$\rho\gamma_u\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=c^2 g_{\mu\nu}U^\mu f^\nu \ , \tag{46}$$

the internal energy equation. The operator D/Dt is the ordinary Lagrangean time derivative $\partial/\partial t+\mathbf{u}\cdot\nabla$; the factor γ_u converts it into a derivative with respect to proper time following the fluid. Equation (46) may also be used to eliminate the internal energy changes from the acceleration equation (45), leading to the second form

$$\begin{aligned}\rho\gamma_u\left(1+\frac{e}{c^2}+\frac{p}{\rho c^2}\right)\frac{D}{Dt}(\gamma_u\mathbf{u}) = \\ -\nabla p+\mathbf{f}-\frac{\gamma_u}{c^2}\left(c^2 g_{\mu\nu}U^\mu f^\nu+\gamma_u\frac{Dp}{Dt}\right)\mathbf{u}\ .\end{aligned} \tag{47}$$

4.1. Coupling to radiation

The interaction of radiation and matter provides a 4-force f^μ. In fact, f^μ is just the negative of the right-hand side of the radiation stress-energy conservation law, (36),

$$f^\mu=-e^\mu_d\frac{1}{c^2}\int d\nu_0 d\Omega\left(1 \quad c\mathbf{n}_0\right)^d (j^0_{\nu_0}-k^0_{\nu_0}I^0)\ , \tag{48}$$

so, in particular,

$$\begin{aligned} f^0 &= -\frac{1}{c^2}\int d\nu_0 d\Omega\,\gamma_u(1+\mathbf{n}_0\cdot\mathbf{u}/c))(j^0_{\nu_0}-k^0_{\nu_0}I^0) \\ \mathbf{f} &= -\frac{1}{c^2}\int d\nu_0 d\Omega\,(c\mathbf{n}_0+\gamma_u\mathbf{u}+(\gamma_u-1)c(\mathbf{n}_0\cdot\mathbf{u})\,\mathbf{u}/u^2)(j^0_{\nu_0}-k^0_{\nu_0}I^0)\ .\end{aligned} \tag{49}$$

The internal energy equation (46) is especially simple, since the dot product of U^μ with e^μ_d produces δ_{d0}:

$$\rho\gamma_u\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=-\int d\nu_0 d\Omega\,(j^0_{\nu_0}-k^0_{\nu_0}I^0)\ . \tag{50}$$

In the notation of Mihalas and Mihalas [6], equations (49) can be written in terms of the components of the g^μ 4-vector,

$$f^0 = -\frac{1}{c^2}g^0=-\frac{1}{c^2}\gamma_u\left(g^0_0+\mathbf{u}\cdot\mathbf{g}_0/c^2\right) \tag{51}$$

$$\mathbf{f} = -\frac{1}{c^2}\mathbf{g}=-\frac{1}{c^2}\left[\mathbf{g}_0+\gamma_u g^0_0\mathbf{u}+\frac{\gamma_u-1}{u^2}(\mathbf{g}_0\cdot\mathbf{u})\mathbf{u}\right]\ , \tag{52}$$

with the fluid-frame components given by

$$g_0^0 = \int d\nu_0 d\Omega\, (j^0_{\nu_0} - k^0_{\nu_0} I^0) \quad \text{and} \quad \mathbf{g}_0 = \int d\nu_0 d\Omega\, c\mathbf{n}_0 (j^0_{\nu_0} - k^0_{\nu_0} I^0)\ . \tag{53}$$

Thus the right-hand side of the internal energy equation is $-g_0^0$, an exact relation. In the acceleration equation (47), the correction term to $\mathbf{f}$ involving $g_{\mu\nu} U^\mu f^\nu$ is the part of the difference between $\mathbf{f}_0$ and $\mathbf{f}$ that is linear in $\mathbf{u}$. This removes a mysterious $O(u/c)$ phenomenon in NR hydro coupled to exact radiation transport, which is that the radiative heating rate introduces a pseudo-force opposite to the velocity. This is cancelled to $O(u/c)$ by allowing the inertial mass to increase with the enthalpy; *i.e.*, with relativistic hydrodynamics. The offsetting $(Dp/Dt)\mathbf{u}$ term in the acceleration equation (47) is also, to $O(u/c)$, equal to the difference between ∇p in the fluid frame and in the lab frame that is due to the Lorentz transformation, so that to that order, it is the fluid-frame pressure gradient that accelerates the fluid.

ACKNOWLEDGMENTS

It is an honor to present this work as part of the celebration of Dimitri Mihalas's 70th birthday. In radiation hydrodynamics as in the several other fields discussed in the workshop, Dimitri made the definitive contribution, in this case of *Foundations of Radiation Hydrodynamics*. One day the workers in applications of radiation hydrodynamics will be fully able to rise to the challenge of that massive volume.

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