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Decycling Numbers of Strong Product Graphs involving Paths, Circuits, Stars or Complete Graphs

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ABSTRACT

In this article, we studied these strong product graphs involving $P_m \boxtimes C_n$, $C_m \boxtimes C_n$, $K_m \boxtimes P_n$, $K_m \boxtimes C_n$, $K_m \boxtimes K_n$, $K_{1,m-1} \boxtimes P_n$ *and* $K_{1,m-1} \boxtimes C_n$. *We have found the decycling numbers of those graphs based on a sandwich principle.*

Keywords: decycling number, strong product, path, circuit, star, complete graph.

2000 Mathematics Subject Classification: 05C38, 68R10, 94C15.

1 Introduction

The problem of eliminating all cycles in a graph by removing a set of vertices goes back to the work of Kirchhoff in [1] on spanning trees. The problem of determining decycling numbers has long been known to be NP-complete, as shown in [2]. Therefore, many results on exact values of decycling numbers have been obtained only for some special families of graphs, such as cubes and grids in [3] and [4]. The exact value of decycling numbers of cartesian product of two cycles has been obtained in [5]. And in [6], the exact value of decycling numbers of strong product of two paths has been obtained. It has also been shown in [6] that there exist a sharp lower bound and a sharp upper bound for the decycling number of a strong product graph $G_1 \boxtimes G_2$. Various other results can be found in [7].

In this article, we are focused on the decycling numbers of the strong product graphs. The outline of this paper is as follows: We first describe the problem and introduce already-known theorems and lemmas in Section 2. Some new notations and new lemmas are then given in Section 3. Finally, we propose our main results that contains seven crucial theorems in Section 4.

2 The problem and past results

The graphs considered here are finite, simple, undirected and denoted by $G = (V(G), E(G))$. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. A *strong product* of two graphs G_1 with G_2 is thus defined as $G_1 \boxtimes G_2 = (V, E)$ where $V = \{(u_i, v_j) \mid \forall\ u_i \in V_1,\ v_j \in V_2\}$, $E = \{((u_1, v_1), (u_2, v_2)) \mid \forall\ (u_1, v_1),\ (u_2, v_2) \in V,\ (u_1, u_2) \in E_1$ when $v_1 = v_2$ or $(v_1, v_2) \in E_2$ when $u_1 = u_2$ or $(u_1, u_2) \in E_1$ and $(v_1, v_2) \in E_2\}$. If $S \subseteq V(G)$ and $G - S$ is acyclic, then S is a *decycling set* of G. A decycling set of the smallest size is a *minimum decycling set* and denoted by ϕ-set. The size of a ϕ-set of G is the *decycling number* of G and denoted by $\phi(G)$. Please refer to [8] for the corresponding definitions and notations.

Theorem 2.1 [4]. *If G and H are homeomorphic graphs, then $\phi(G) = \phi(H)$.*

Lemma 2.2 [6]. *If $G_1 \subseteq G,\ G_2 \subseteq G$ and $V(G_1) \cap V(G_2) = \emptyset$, then $\phi(G) \geq \phi(G_1) + \phi(G_2)$.*

Lemma 2.3 [6]. $\phi(G_1 \boxtimes G_2) \geq \max\{|G_1| \cdot \phi(G_2),\ |G_2| \cdot \phi(G_1)\}$.

Lemma 2.4 [6]. *Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$. For each $V_1' \subseteq V_1,\ V_2' \subseteq V_2$,* $\phi(G_1 \boxtimes G_2) \geq \max\{\phi(G_1|_{V_1'} \boxtimes G_2) + \phi(G_1|_{V_1 - V_1'} \boxtimes G_2),\ \phi(G_1 \boxtimes G_2|_{V_2'}) + \phi(G_1 \boxtimes G_2|_{V_2 - V_2'})\}$.

Lemma 2.5 [6]. $\phi(G_1 \boxtimes G_2) \leq \min\{|G_1| \cdot |G_2| - \alpha(G_1) \cdot (|G_2| - \phi(G_2)),\ |G_1| \cdot |G_2| - \alpha(G_2) \cdot (|G_1| - \phi(G_1))\}$, *where $\alpha(G)$ denotes the independence number of graph G.*

Lemma 2.6 [6]. $\phi(P_2 \boxtimes P_n) = 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor$.

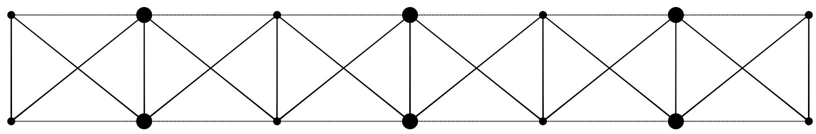

Figure 2.1. A ϕ-set for $P_2 \boxtimes P_7$.

Lemma 2.7 [6]. *The ϕ-set of $P_2 \boxtimes P_{2k+1}$ is $\{\ x_{1,2}, x_{2,2}, x_{1,4}, x_{2,4}, \cdots, x_{1,2k}, x_{2,2k}\ \}$ (see Figure 2.1 for the case $k = 3$).*

Theorem 2.8 [6]. $\phi(P_m \boxtimes P_n) = \min\{m \cdot \left\lfloor \frac{n}{2} \right\rfloor,\ n \cdot \left\lfloor \frac{m}{2} \right\rfloor\}$.

3 Notations and Preliminaries

In order to prove the following new results, we introduce some useful notations as follows. Let the vertex sets of G_1 and G_2 be $\{u_1, u_2, \cdots, u_{|G_1|}\}$ and $\{v_1, v_2, \cdots, v_{|G_2|}\}$ respectively. Denote $x_{i,j} = (u_i, v_j)$ and $R(i) = \{x_{i,j} \mid j = 1, 2, \cdots, |G_2|\}$ for $i = 1, 2, \cdots, |G_1|$ and $C(j) = \{x_{i,j} \mid i = 1, 2, \cdots, |G_1|\}$ for $j = 1, 2, \cdots, |G_2|$. $R(i)$ is called the i-th *row* and $C(j)$ is called the j-th *column* of $G_1 \boxtimes G_2$. If S is a vertex subset of $G_1 \boxtimes G_2$, we denote the vertices of S in the i-th row of $G_1 \boxtimes G_2$ by $S(R(i))$ and the vertices of S in the j-th column of $G_1 \boxtimes G_2$ by $S(C(j))$, and by extension, for $k \leq l$, denote the set $S(R(k)) \cup S(R(k+1)) \cup \cdots \cup S(R(l))$ by $S(R(k, l))$ and the set $S(C(k)) \cup S(C(k+1)) \cup \cdots \cup S(C(l))$ by $S(C(k, l))$.

In the following P_n, C_n, $K_{1,\, n-1}$ and K_n denotes a path, circuit, star and complete graph on n vertices respectively. Especially, in the following we denote a vertex in a decycling set by a bigger black dot " • ". For our purpose we need the following statement.

Lemma 3.1. $\phi(P_2 \boxtimes C_n) = 2 \cdot \left\lceil \frac{n}{2} \right\rceil$.

Proof. We first prove that there is no ϕ-set of $P_2 \boxtimes C_n$ containing at most one vertex from each column of $P_2 \boxtimes C_n$. If $S \subseteq P_2 \boxtimes C_n$ and $|S(C(j))| \leq 1$ for $j = 1, 2, \cdots, n$, then $P_2 \boxtimes C_n - S$ has a subgraph isomorphic to C_n. Therefore, there is no ϕ-set of $P_2 \boxtimes C_n$ containing at most one vertex for each column of $P_2 \boxtimes C_n$. That is, there always exists a column $C(j)$ $(1 \leq j \leq n)$ of $P_2 \boxtimes C_n$ so that $|S(C(j))| = 2$. Without loss of generality, we assume that $|S(C(1))| = 2$. Then, by Lemma 2.6, $|S| = |S(C(1))| + |S(C(2, n))| \geq 2 + \phi(P_2 \boxtimes P_{n-1}) = 2 + 2 \cdot \lfloor \frac{n-1}{2} \rfloor = 2 + 2 \cdot \lceil \frac{n-2}{2} \rceil = 2 \cdot \lceil \frac{n}{2} \rceil$. On the other hand, there are decycling sets of this size, such as $\{x_{1,1}, x_{2,1}, x_{1,3}, x_{2,3}, \cdots, x_{1,2k-1}, x_{2,2k-1}\}$ where $k = \lceil \frac{n}{2} \rceil$. So the lemma follows (see Figure 3.1 for the case $P_2 \boxtimes C_6$). □

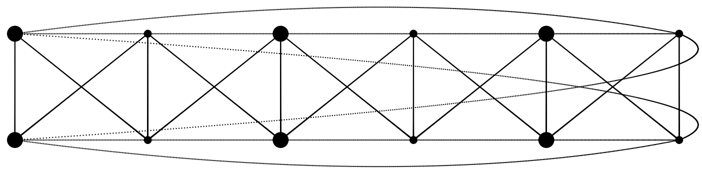

Figure 3.1. A ϕ-set for $P_2 \boxtimes C_6$.

Lemma 3.2. *The ϕ-set of $P_2 \boxtimes C_{2k}$ is $\{x_{1,1}, x_{2,1}, x_{1,3}, x_{2,3}, \cdots, x_{1,2k-1}, x_{2,2k-1}\}$ or $\{x_{1,2}, x_{2,2}, x_{1,4}, x_{2,4}, \cdots, x_{1,2k}, x_{2,2k}\}$.*

Proof. By the proof of Lemma 3.1, for any ϕ-set S of $P_2 \boxtimes C_{2k}$, there exists a column $C(j)$ $(1 \leq j \leq 2k)$ of $P_2 \boxtimes C_{2k}$ so that $|S(C(j))| = 2$. Without loss of generality, we assume that $|S(C(1))| = 2$. Then $P_2 \boxtimes C_{2k} - S(C(1))$ is a copy of $P_2 \boxtimes P_{2k-1}$. By Lemma 2.7, the ϕ-set of $P_2 \boxtimes C_{2k} - S(C(1))$ is $\{x_{1,3}, x_{2,3}, \cdots, x_{1,2k-1}, x_{2,2k-1}\}$. Hence $S = \{x_{1,1}, x_{2,1}, x_{1,3}, x_{2,3}, \cdots, x_{1,2k-1}, x_{2,2k-1}\}$. By symmetry, $\{x_{1,2}, x_{2,2}, x_{1,4}, x_{2,4}, \cdots, x_{1,2k}, x_{2,2k}\}$ is the other ϕ-set of $P_2 \boxtimes C_{2k}$, which proves the lemma (see Figure 3.1 for a ϕ-set of $P_2 \boxtimes C_6$). □

Lemma 3.3. $\phi(K_m \boxtimes P_2) = 2 \cdot m - 2$.

Proof. From the definition of the strong product of two graphs, we can conclude that $K_m \boxtimes P_2$ is isomorphic to K_{2m}. Thus $\phi(K_m \boxtimes P_2) = \phi(K_{2m}) = 2 \cdot m - 2$. □

4 Main results

In this section, we present our seven new theorems.

Theorem 4.1. $\phi(P_m \boxtimes C_n) = \min\{m \cdot \lceil \frac{n}{2} \rceil,\ n \cdot \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil\}$.

Proof. Applying Lemma 2.5, $\phi(P_m \boxtimes C_n) \leq \min\{m \cdot n - \lceil \frac{m}{2} \rceil \cdot (n-1),\ m \cdot n - \lfloor \frac{n}{2} \rfloor \cdot m\} = \min\{n \cdot \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil,\ m \cdot \lceil \frac{n}{2} \rceil\} = \min\{m \cdot \lceil \frac{n}{2} \rceil,\ n \cdot \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil\}$. So in the rest of the proof, we only have to prove that $\phi(P_m \boxtimes C_n) \geq \min\{m \cdot \lceil \frac{n}{2} \rceil,\ n \cdot \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil\}$. Consider the following four cases.

Case 1. $m = 2k$.

Since $m = 2k$, hence $\min\{m \cdot \lceil \frac{n}{2} \rceil,\ n \cdot \lfloor \frac{m}{2} \rfloor + \lceil \frac{m}{2} \rceil\} = m \cdot \lceil \frac{n}{2} \rceil$. So it suffices to prove that $\phi(P_m \boxtimes C_n) \geq m \cdot \lceil \frac{n}{2} \rceil$. We use induction on m. When $m = 2$, by Lemma 3.1, the inequality holds. Now we assume that for $m = 2(k-1)$, the inequality holds. Let $m = 2k$. By Lemma 2.4 and the induction hypothesis, $\phi(P_m \boxtimes C_n) \geq \phi(P_2 \boxtimes C_n) + \phi(P_{m-2} \boxtimes C_n) = 2 \cdot \lceil \frac{n}{2} \rceil + (m-2) \cdot \lceil \frac{n}{2} \rceil = m \cdot \lceil \frac{n}{2} \rceil$.

Case 2. $m = 2k+1,\ n = 2s+1$.

Since $m = 2k+1,\ n = 2s+1$, hence $\min\{m\cdot\lceil\frac{n}{2}\rceil,\ n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil\} = n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$. So we only have to prove that $\phi(P_m\boxtimes C_n)\geq n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$. We use induction on m. When $m=1$, the inequality obviously holds. Now we assume that for $m=2(k-1)+1$, the inequality holds. Let $m=2k+1$. By Lemma 2.4 and the induction hypothesis, $\phi(P_m\boxtimes C_n)\geq\phi(P_2\boxtimes C_n)+\phi(P_{m-2}\boxtimes C_n)\geq 2\cdot\lceil\frac{n}{2}\rceil+n\cdot\lfloor\frac{m-2}{2}\rfloor+\lceil\frac{m-2}{2}\rceil = n+1+n\cdot\lfloor\frac{m}{2}\rfloor-n+\lceil\frac{m}{2}\rceil-1 = n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$.

Case 3. $m=2k+1,\ n=2s\ (k<s)$.

Since $m=2k+1,\ n=2s\ (k<s)$, hence $\min\{m\cdot\lceil\frac{n}{2}\rceil,\ n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil\} = n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$. So it suffices to prove that $\phi(P_m\boxtimes C_n)\geq n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$. let S be any ϕ-set of $P_m\boxtimes C_n$. It is obvious that for any $i\ (1\leq i\leq m-1)$, $|S(R(i))|\geq 1$ and $|S(R(i,i+1))|\geq\phi(P_2\boxtimes C_n)=n$. Now if for any $i\ (1\leq i\leq m-1)$, $|S(R(i,i+1))|\geq\phi(P_2\boxtimes C_n)+1=n+1$, then we have $|S|=|S(R(1))|+|S(R(2,3))|+|S(R(4,5))|+\cdots+|S(R(2k,2k{+}1))|\geq 1+k\cdot(n{+}1)=1+\lfloor\frac{m}{2}\rfloor\cdot(n{+}1)=n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$. Otherwise, there exists some $i\ (1\leq i\leq m-1)$ so that $|S(R(i,i+1))|=\phi(P_2\boxtimes C_n)=n$. According to Lemma 3.2, we know that $|S(R(i))|=|S(R(i+1))|=\frac{n}{2}$. When i is odd, by Case 1, $|S|=|S(R(1,i-1))|+|S(R(i))|+|S(R(i+1,m))|\geq\phi(P_{i-1}\boxtimes C_n)+\frac{n}{2}+\phi(P_{m-i}\boxtimes C_n)\geq(i-1)\cdot\lceil\frac{n}{2}\rceil+\lceil\frac{n}{2}\rceil+(m-i)\cdot\lceil\frac{n}{2}\rceil=m\cdot\lceil\frac{n}{2}\rceil\geq n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$. When i is even, again by Case 1, $|S|=|S(R(1,i))|+|S(R(i+1))|+|S(R(i+2,m))|\geq\phi(P_i\boxtimes C_n)+\frac{n}{2}+\phi(P_{m-i-1}\boxtimes C_n)\geq i\cdot\lceil\frac{n}{2}\rceil+\lceil\frac{n}{2}\rceil+(m-i-1)\cdot\lceil\frac{n}{2}\rceil=m\cdot\lceil\frac{n}{2}\rceil\geq n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil$.

Case 4. $m=2k+1,\ n=2s\ (k\geq s)$.

Since $m=2k+1,\ n=2s\ (k\geq s)$, hence $\min\{m\cdot\lceil\frac{n}{2}\rceil,\ n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil\}=m\cdot\lceil\frac{n}{2}\rceil$. So we only have to prove that $\phi(P_m\boxtimes C_n)\geq m\cdot\lceil\frac{n}{2}\rceil$. Let S be any ϕ-set of $P_m\boxtimes C_n$. It is obvious that for any $i\ (1\leq i\leq m-1)$, $|S(R(i))|\geq 1$ and $|S(R(i,i+1))|\geq\phi(P_2\boxtimes C_n)=n$. Now if for any $i\ (1\leq i\leq m-1)$, $|S(R(i,i+1))|\geq\phi(P_2\boxtimes C_n)+1=n+1$, then we can get that $|S|=|S(R(1))|+|S(R(2,3))|+|S(R(4,5))|+\cdots+|S(R(2k,2k+1))|\geq 1+k\cdot(n+1)=1+\lfloor\frac{m}{2}\rfloor\cdot(n+1)=n\cdot\lfloor\frac{m}{2}\rfloor+\lceil\frac{m}{2}\rceil\geq m\cdot\lceil\frac{n}{2}\rceil$. Otherwise, there exists some $i\ (1\leq i\leq m-1)$ so that $|S(R(i,i+1))|=\phi(P_2\boxtimes C_n)=n$. According to Lemma 3.2, we know that $|S(R(i))|=|S(R(i{+}1))|=\frac{n}{2}$. When i is odd, by Case 1, $|S|=|S(R(1,i{-}1))|+|S(R(i))|+|S(R(i{+}1,m))|\geq\phi(P_{i-1}\boxtimes C_n)+\frac{n}{2}+\phi(P_{m-i}\boxtimes C_n)\geq(i{-}1)\cdot\lceil\frac{n}{2}\rceil+\lceil\frac{n}{2}\rceil+(m{-}i)\cdot\lceil\frac{n}{2}\rceil=m\cdot\lceil\frac{n}{2}\rceil$. When i is even, again by Case 1, $|S|=|S(R(1,i))|+|S(R(i{+}1))|+|S(R(i{+}2,m))|\geq\phi(P_i\boxtimes C_n)+\frac{n}{2}+\phi(P_{m-i-1}\boxtimes C_n)\geq i\cdot\lceil\frac{n}{2}\rceil+\lceil\frac{n}{2}\rceil+(m-i-1)\cdot\lceil\frac{n}{2}\rceil=m\cdot\lceil\frac{n}{2}\rceil$. This completes the proof of Theorem 4.1. □

Theorem 4.2. *When m or n is even, $\phi(C_m\boxtimes C_n)=\min\{m\cdot\lceil\frac{n}{2}\rceil+\lfloor\frac{n}{2}\rfloor,\ n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor\}$; when m and n are both odd and $m\leq n$, $n\cdot\lceil\frac{m}{2}\rceil\leq\phi(C_m\boxtimes C_n)\leq n\cdot\lceil\frac{m}{2}\rceil+1$.*

Proof. When m or n is even. Applying Lemma 2.5, $\phi(C_m\boxtimes C_n)\leq\min\{m\cdot n-\lfloor\frac{m}{2}\rfloor\cdot(n-1),\ m\cdot n-\lfloor\frac{n}{2}\rfloor\cdot(m-1)\}=\min\{n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor,\ m\cdot\lceil\frac{n}{2}\rceil+\lfloor\frac{n}{2}\rfloor\}=\min\{m\cdot\lceil\frac{n}{2}\rceil+\lfloor\frac{n}{2}\rfloor,\ n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor\}$. So in the rest of the proof, we only have to prove that $\phi(C_m\boxtimes C_n)\geq\min\{m\cdot\lceil\frac{n}{2}\rceil+\lfloor\frac{n}{2}\rfloor,\ n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor\}$. Since $\phi(C_m\boxtimes C_n)=\phi(C_n\boxtimes C_m)$, without loss of generality, we can assume that $m\leq n$. So it suffices to prove the following three cases.

Case 1. $m=2k,\ n=2s+1\ (k\leq s)$.

Since $m=2k,\ n=2s+1\ (k\leq s)$, hence $\min\{m\cdot\lceil\frac{n}{2}\rceil+\lfloor\frac{n}{2}\rfloor,\ n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor\}=n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor$. So we only have to prove that $\phi(C_m\boxtimes C_n)\geq n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor$. Since $\phi(P_2\boxtimes C_n)=2\cdot\lceil\frac{n}{2}\rceil=n+1$, hence $\phi(C_m\boxtimes C_n)\geq\phi(P_m\boxtimes C_n)\geq k\cdot\phi(P_2\boxtimes C_n)=k\cdot(n+1)=\frac{m}{2}\cdot(n+1)=n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor$.

Case 2. $m=2k+1,\ n=2s\ (k<s)$.

Since $m=2k+1,\ n=2s\ (k<s)$, hence $\min\{m\cdot\lceil\frac{n}{2}\rceil+\lfloor\frac{n}{2}\rfloor,\ n\cdot\lceil\frac{m}{2}\rceil+\lfloor\frac{m}{2}\rfloor\}=m\cdot\lceil\frac{n}{2}\rceil+\lfloor\frac{n}{2}\rfloor$.

So it suffices to prove that $\phi(C_m \boxtimes C_n) \geq m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$. Since $\phi(C_m \boxtimes P_2) = 2 \cdot \lceil \frac{m}{2} \rceil = m+1$, hence $\phi(C_m \boxtimes C_n) \geq \phi(C_m \boxtimes P_n) \geq s \cdot \phi(C_m \boxtimes P_2) = s \cdot (m+1) = \frac{n}{2} \cdot (m+1) = m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$.
Case 3. $m = 2k,\ n = 2s\ (k \leq s)$.
Since $m = 2k,\ n = 2s\ (k \leq s)$, hence $\min\{m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor,\ n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor\} = n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. So it suffices to prove that $\phi(C_m \boxtimes C_n) \geq n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. Let S be any ϕ-set of $C_m \boxtimes C_n$. By Lemma 3.1, we know that for any i $(1 \leq i \leq m$ when $i = m,\ i+1 = 1)$, $|S(R(i,i+1))| \geq \phi(P_2 \boxtimes C_n) = n$. Now if for any i, $|S(R(i,i+1))| \geq \phi(P_2 \boxtimes C_n) + 1 = n+1$, then we can get that $|S| = |S(R(1,2))| + |S(R(3,4))| + \cdots + |S(R(2k-1,2k))| \geq k \cdot (n+1) = \frac{m}{2} \cdot (n+1) = n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$. Otherwise, there exists some i so that $|S(R(i,i+1))| = \phi(P_2 \boxtimes C_n) = n$. Without loss of generality, we assume that $|S(R(1,2))| = \phi(P_2 \boxtimes C_n) = n$. By Lemma 3.2 and symmetry, $S(R(1,2)) = \{x_{1,1}, x_{2,1}, x_{1,3}, x_{2,3}, \cdots, x_{1,n-1}, x_{2,n-1}\}$. Hence $\{x_{1,1}, x_{2,1}\} \subset S(C(1,2))$. Again by Lemma 3.2, we conclude that $|S(C(1,2))| \geq m+1$. Using the same method, we know that $|S(C(3,4))| = \cdots = |S(C(n-1,n))| \geq m+1$. Hence $|S| = |S(C(1,2))| + |S(C(3,4))| + \cdots + |S(C(n-1,n))| \geq s \cdot (m+1) = \frac{n}{2} \cdot (m+1) = m \cdot \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor \geq n \cdot \lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor$.
When m and n are both odd and $m \leq n$. Let S be any ϕ-set of $C_m \boxtimes C_n$. By Lemma 3.1, we know that for any j $(1 \leq j \leq n$ when $j = n,\ j+1 = 1)$, $|S(C(j,j+1))| \geq \phi(C_m \boxtimes P_2) = 2 \cdot \lceil \frac{m}{2} \rceil$. Hence, there exists some j $(1 \leq j \leq n)$ so that $|S(C(j))| \geq \lceil \frac{m}{2} \rceil$. Without loss of generality, we assume that $|S(C(1))| \geq \lceil \frac{m}{2} \rceil$. Then again by Lemma 3.1, $|S| = |S(C(1))| + |S(C(2,3))| + \cdots + |S(C(n-1,n))| \geq \lceil \frac{m}{2} \rceil + \frac{n-1}{2} \cdot 2 \cdot \lceil \frac{m}{2} \rceil = n \cdot \lceil \frac{m}{2} \rceil$. Next we construct a decycling set S' of $C_m \boxtimes C_n$ as follow:
$S' = S'(C(1)) \cup S'(C(2)) \cup \cdots \cup S'(C(n))$,
$S'(C(1)) = \{x_{1,1}, x_{\lceil \frac{m}{2} \rceil,1}, x_{\lceil \frac{m}{2} \rceil+1,1}, \cdots, x_{m-1,1}, x_{m,1}\}$,
$S'(C(i)) = \{x_{f(1+i\cdot\lfloor \frac{m}{2} \rfloor),i}, x_{f(2+i\cdot\lfloor \frac{m}{2} \rfloor),i}, \cdots, x_{f(\lfloor \frac{m}{2} \rfloor+i\cdot\lfloor \frac{m}{2} \rfloor),i}, x_{f(\lceil \frac{m}{2} \rceil+i\cdot\lfloor \frac{m}{2} \rfloor),i}\} \qquad (2 \leq i \leq m)$,

$$\text{where} \quad f(x) = \begin{cases} \bmod\ (x,m) & m \nmid x, \\ m & m \mid x. \end{cases}$$

$S'(C(i)) = \{x_{\lceil \frac{m}{2} \rceil,i}, x_{\lceil \frac{m}{2} \rceil+1,i}, \cdots, x_{m-1,i}, x_{m,i}\} \qquad (m+1 \leq i \leq n$ and $\bmod\ (i,2) = 0)$,
$S'(C(i)) = \{x_{1,i}, x_{2,i}, \cdots, x_{\lfloor \frac{m}{2} \rfloor,i}, x_{\lceil \frac{m}{2} \rceil,i}\} \qquad (m+1 \leq i \leq n$ and $\bmod\ (i,2) = 1)$.
On one hand, since every column of $C_m \boxtimes C_n$ contains $\lceil \frac{m}{2} \rceil$ vertices of S' except of the first column, which contains $\lceil \frac{m}{2} \rceil + 1$ vertices of S', we infer that $|S'| = n \cdot \lceil \frac{m}{2} \rceil + 1$. On the other hand,it is easy to find that $C_m \boxtimes C_n - S'$ is a tree. Hence $\phi(C_m \boxtimes C_n) \leq n \cdot \lceil \frac{m}{2} \rceil + 1$. So $n \cdot \lceil \frac{m}{2} \rceil \leq \phi(C_m \boxtimes C_n) \leq n \cdot \lceil \frac{m}{2} \rceil + 1$. This completes the proof of Theorem 4.2. □

Theorem 4.3. $\phi(K_m \boxtimes P_n) = m \cdot n - 2 \cdot \lceil \frac{n}{2} \rceil$.

Proof. Applying Lemma 2.5, $\phi(K_m \boxtimes P_n) \leq \min\{m \cdot n - n,\ m \cdot n - \lceil \frac{n}{2} \rceil \cdot 2\} = m \cdot n - 2 \cdot \lceil \frac{n}{2} \rceil$. So in the rest of the proof, we only have to prove that $\phi(K_m \boxtimes P_n) \geq m \cdot n - 2 \cdot \lceil \frac{n}{2} \rceil$. We use induction on n. When $n = 1$, $\phi(K_m \boxtimes P_n) = \phi(K_m) = m-2$, the inequality holds. When $n = 2$, by Lemma 3.3, the inequality holds. Now we assume that for $n = s (s \geq 2)$, the inequality holds. Let $n = s+2$. By Lemma 2.4 and the induction hypothesis, $\phi(K_m \boxtimes P_n) = \phi(K_m \boxtimes P_{s+2}) \geq \phi(K_m \boxtimes P_s) + \phi(K_m \boxtimes P_2) \geq m \cdot s - 2 \cdot \lceil \frac{s}{2} \rceil + 2 \cdot m - 2 = m(s+2) - 2 \cdot \lceil \frac{s+2}{2} \rceil = m \cdot n - 2 \cdot \lceil \frac{n}{2} \rceil$. So for any n, the inequality holds. Hence this theorem has been proved. □

Theorem 4.4. $\phi(K_m \boxtimes C_n) = m \cdot n - 2 \cdot \lfloor \frac{n}{2} \rfloor$.

Proof. Applying Lemma 2.5, $\phi(K_m \boxtimes C_n) \leq \min\{m \cdot n - (n-1),\ m \cdot n - \lfloor \frac{n}{2} \rfloor \cdot 2\} = m \cdot n - 2 \cdot \lfloor \frac{n}{2} \rfloor$.

So in the rest of the proof, we only have to prove that $\phi(K_m \boxtimes C_n) \geq m \cdot n - 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor$. Consider the following two cases.
Case 1. $n = 2s$.
Since $n = 2s$, hence $m \cdot n - 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor = 2 \cdot m \cdot s - 2 \cdot s = (m-1) \cdot n$. So we only have to prove that $\phi(K_m \boxtimes C_n) \geq (m-1) \cdot n$. By Lemma 2.2 and Theorem 4.3, $\phi(K_m \boxtimes C_n) \geq \phi(K_m \boxtimes P_n) = m \cdot n - 2 \cdot \left\lceil \frac{n}{2} \right\rceil = m \cdot n - n = (m-1) \cdot n$.
Case 2. $n = 2s+1$.
Since $n = 2s+1$, hence $m \cdot n - 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor = m \cdot n - (n-1) = m \cdot n - n + 1$. So it suffices to prove that $\phi(K_m \boxtimes C_n) \geq m \cdot n - n + 1$. Let S be any ϕ-set of $K_m \boxtimes C_n$. We can conclude that there exist some column $C(j)$ $(1 \leq j \leq n)$ of $K_m \boxtimes C_n$, so that $|S(C(j))| = m$. Otherwise, for any column of $K_m \boxtimes C_n$, there exist at least one vertex in graph $K_m \boxtimes C_n - S$. Then $K_m \boxtimes C_n - S$ has a subgraph isomorphic to C_n, which contradicts the fact that S is a ϕ-set of $K_m \boxtimes C_n$. So without loss of generality, we assume that $|S(C(1))| = m$. Furthermore, for any i $(2 \leq i \geq n-1)$, by Lemma 3.3, we can conclude that $|S(C(i, i+1))| \geq \phi(K_m \boxtimes P_2) = 2 \cdot m - 2$. So $|S| = |S(C(1))| + |S(C(2,3))| + \cdots + |S(C(2s, 2s+1))| \geq m + s \cdot (2 \cdot m - 2) = m + 2 \cdot s \cdot m - 2 \cdot s = m + (n-1) \cdot m - (n-1) = m \cdot n - n + 1$. Hence this proves the theorem. □

Theorem 4.5. $\phi(K_m \boxtimes K_n) = m \cdot n - 2$.

Proof. From the definition of the strong product of two graphs, we can conclude that $K_m \boxtimes K_n$ is isomorphic to K_{mn}. Thus $\phi(K_m \boxtimes K_n) = \phi(K_{mn}) = m \cdot n - 2$. □

Theorem 4.6. *When* $m \geq 3$*,* $\phi(K_{1,m-1} \boxtimes P_n) = n$*.*

Proof. When $m \geq 3$, we can conclude that $K_{1,\,m-1} \boxtimes P_n \supseteq P_3 \boxtimes P_n$. By Lemma 2.2 and Theorem 2.8, $\phi(K_{1,m-1} \boxtimes P_n) \geq \phi(P_3 \boxtimes P_n) = \min\{3 \cdot \left\lfloor \frac{n}{2} \right\rfloor,\ n \cdot \left\lfloor \frac{3}{2} \right\rfloor\} = n$. On the other hand, from the definition of strong product of two graphs, we know that for $\forall j \in 1, 2, \cdots, n$, the induced subgraph $(K_{1,m-1} \boxtimes P_n)|_{C(j)}$ is a copy of $K_{1,m-1}$. If we denotes the vertex of maximum degree in $(K_{1,m-1} \boxtimes P_n)|_{C(j)}$ by $x_{1,j}$, then vertex set $S = \{x_{1,1}, x_{1,2}, \cdots, x_{1,n}\}$ is a decycling set of $K_{1,m-1} \boxtimes P_n$ where $|S| = n$. □

Theorem 4.7. *When* $m,\ n \geq 3$*,* $\phi(K_{1,m-1} \boxtimes C_n) = n + m - 1$*.*

Proof. When $m,\ n \geq 3$, we denote $K_{1,m-1} = \{u_1, u_2, \cdots, u_m\}$ where u_1 is the vertex of maximum degree in $K_{1,m-1}$, denote $C_n = v_1 v_2 \cdots v_n v_1$ and denote (u_i, v_j) in $K_{1,m-1} \boxtimes C_n$ by $x_{i,j}$. We first prove that $\phi(K_{1,m-1} \boxtimes C_n) \geq n + m - 1$. Now let S be any ϕ-set of $K_{1,m-1} \boxtimes C_n$. So it suffices to prove that $|S| \geq n + m - 1$. Here we only have to prove the following three cases.
Case 1. $|S(R(1))| = n$.
Since $|S(R(1))| = n$, hence for any i $(2 \leq i \leq m)$, $|S(R(i))| \geq 1$. Otherwise, there exists some i $(2 \leq i \leq m)$, $N(R(i)) = 0$. Then circuit $C_n = x_{i,1} x_{i,2} \cdots x_{i,n} x_{i,1}$ is a subgraph of $K_{1,m-1} \boxtimes C_n - S$, contradicting the fact that S is a ϕ-set of $K_{1,m-1} \boxtimes C_n$. Hence $|S| = |S(R(1))| + |S(R(2))| + \cdots + |S(R(m))| \geq n + (m-1) \cdot 1 = n + m - 1$.
Case 2. $|S(R(1))| = n - 1$.
Since $|S(R(1))| = n - 1$, there exists some j $(1 \leq j \leq n)$ so that $x_{1,j} \notin S$. By symmetry, we can assume that $x_{1,1} \notin S$. Then for any i $(2 \leq i \leq m)$, $K_{1,m-1} \boxtimes C_n|_{\{x_{1,1}, x_{i,1}, x_{i,2}, \cdots, x_{i,n}\}}$ is homeomorphic to K_4. Hence for any i $(2 \leq i \leq m)$, $|S(R(i))| \geq 2$. So, for $m \geq 3$, $|S| = |S(R(1))| + |S(R(2))| + \cdots + |S(R(m))| \geq (n-1) + (m-1) \cdot 2 = n + m - 1 + (m-2) \geq n + m - 1$.

Case 3. $|S(R(1))| = n - k\ (2 \leq k \leq n)$.
By Lemma 3.1, $\phi(P_2 \boxtimes C_n) = 2 \cdot \lceil \frac{n}{2} \rceil$. Since $|S(R(1))| = n-k$, for any $i\ (2 \leq i \leq m)$, $|S(R(i))| \geq 2 \cdot \lceil \frac{n}{2} \rceil - (n-k) \geq n-(n-k) = k$. Because $m \geq 3$ and $2 \leq k \leq n$, $(m-2)(k-1)-1 \geq 0$. Hence $|S| = |S(R(1))| + |S(R(2))| + \cdots + |S(R(m))| \geq (n-k)+(m-1) \cdot k = n+m-1+(m-2)(k-1)-1 \geq n+m-1$.
So $\phi(K_{1,m-1} \boxtimes C_n) \geq n+m-1$. On the other hand, $S = \{x_{1,1}, x_{1,2}, \cdots, x_{1,n}\} \cup \{x_{2,1}, x_{3,1}, \cdots, x_{m,1}\}$ is a decycling set of $K_{1,m-1} \boxtimes C_n$ where $|S| = n+m-1$. $\square$

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