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# Summation by Parts Finite Difference Approximations for Seismic and Seismo-Acoustic Computations

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# Summation by Parts Finite Difference Approximations for Seismic and Seismo-Acoustic Computations

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**Abstract** We develop stable finite difference approximations for a multi-physics problem that couples elastic wave propagation in one domain to acoustic wave propagation in another domain. The approximation consists of one finite difference schemes in each domain together with discrete interface conditions that couple the two schemes. The finite difference approximations use summation-by-parts (SBP) operators, that lead to stability of the coupled problem. Furthermore, we develop a new way to enforce boundary conditions for SBP discretizations of first order problems. The new method, which uses ghost points to enforce the boundary conditions, is a flexible alternative to the more established projection and SAT methods.

## 1 Introduction

Near surface seismic events emit both elastic waves traveling in the earth and acoustic waves propagating in the atmosphere. Acoustic waves can also occur because of other events, such as bolides or vulcanic eruptions. Elastic and acoustic waves are recorded by seismographs and by infrasound instruments at various locations around the world. A coupled seismo-acoustic modeling capability is of relevance to many applications in order to analyze and understand seismograms and infrasound recordings.

We will here model seismic wave propagation by the elastic wave equation. Acoustic infrasound will be described by the linearized Euler equations of compressible gas dynamics. The elastic and acoustic domains are coupled by interface conditions that enforce continuity of normal stresses and of normal velocities. We will here develop finite difference discretizations, based on the summation-by-parts

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(SBP) principle, of the elastic equations, the acoustic equations, and the interface conditions, that make the coupled seismo-acoustic problem stable.

In [2], we made use of ghost points to enforce physical boundary conditions on SBP discretizations of the elastic wave equation in second order formulation. For first order hyperbolic PDEs, boundary conditions in the SBP context have traditionally been imposed by either projection or penalty term (SAT). We will here develop ghost point enforced boundary conditions also for SBP discretizations of problems in first order formulation.

## 2 SBP operators

Let $D$ be a standard summation by parts finite difference operator for approximating a first derivative. $D$ can be represented as a real $N$ by $N$ matrix acting on grid functions $u = (u_1,\ u_2,\ u_3,\dots,\ u_N)$. The grid functions are defined on a domain $0 \le x \le 1$, with uniformly distributed grid points $x_j = (j-1)h,\ j = 1,2,\dots,N$, where $h = 1/(N-1)$ is the grid spacing. When ghost points are present they are located at the points $j = 0$ and $j = N+1$. The standard SBP identity,

$$(u,Dv)_h = -(Du,v)_h - u_1v_1 + u_Nv_N, \tag{1}$$

is assumed to hold in a scalar product

$$(u,v)_h = h\sum_{j=1}^{N} \omega_j u_j v_j, \tag{2}$$

where $\omega_j$ are positive weights. We extend the difference operator $D$ to handle ghost points by adding an operator to the first and last row of $D$. The resulting operator, $\widetilde{D}$, can be represented as a rectangular matrix with $N$ rows and $N+2$ columns,

$$\widetilde{D} = \begin{pmatrix} \mathbf{0} & D & \mathbf{0} \end{pmatrix} + \frac{1}{h}\begin{pmatrix} \beta^T \\ \mathbf{0} \\ -\delta^T \end{pmatrix}, \tag{3}$$

$$\beta^T = (\beta_0,\ \beta_1,\ \cdots,\ \beta_r,\ 0,\ \cdots,\ 0) \tag{4}$$

$$\delta^T = (0,\ \cdots,\ 0,\ \delta_r,\ \delta_{r-1},\ \cdots,\ \delta_0) \tag{5}$$

At the first grid point, $(Du)_1$ is replaced by $(Du)_1 + \frac{1}{h}\beta^T\tilde{u}$, where we denote

$$\tilde{u} = (u_0,\ u_1,\ u_2,\ \dots,u_N,\ u_{N+1})^T.$$

Similarly, at the last grid point, the difference approximation is $(Du)_N - \frac{1}{h}\delta^T\tilde{u}$.

**Lemma 1.** *The difference operator $\widetilde{D}$ satisfies the SBP-like identity*

$$(u,\widetilde{D}\tilde{v})_h = -(Du,v)_h - u_1(v_1 - \omega_1\beta^T\tilde{v}) + u_N(v_N - \omega_N\delta^T\tilde{v}). \tag{6}$$

*Proof.* The definition of $\widetilde{D}$ gives

$$\widetilde{D}\tilde{v} = Dv + \frac{1}{h}\mathbf{e}_1(\beta^T\tilde{v}) - \frac{1}{h}\mathbf{e}_N(\delta^T\tilde{v}), \tag{7}$$

where $\mathbf{e}_1 = (1,0,\ldots,0)^T$ and $\mathbf{e}_N = (0,\ldots,0,1)^T$. It therefore follows from (1),

$$(u,\widetilde{D}\tilde{v})_h = -(Du,v)_h - u_1v_1 + u_Nv_N + u_1\omega_1\beta^T\tilde{v} - u_N\omega_N\delta^T\tilde{v},$$

which leads to (6).

To illustrate the usage of (6), we consider the initial boundary value problem

$$u_t + a(x)u_x = 0, \quad 0 \le x \le 1, \quad t > 0, \tag{8}$$
$$u(0,t) = g(t), \quad t > 0, \tag{9}$$

where $a(x)$ is a real-valued function. We assume $a(0) > 0$ and $a(1) > 0$. We can write (8) as

$$u_t = -\frac{1}{2}a(x)u_x - \frac{1}{2}(au)_x + \frac{1}{2}a_xu.$$

Multiplying this equation by $u$ and integrating over $0 \le x \le 1$ gives the estimate

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \frac{1}{2}(u,a_xu) + \frac{1}{2}\left[a(0)u(0,t)^2 - a(1)u(1,t)^2\right] \le \frac{1}{2}\alpha\|u\|^2 + \frac{1}{2}a(0)g(t)^2, \tag{10}$$

where $\alpha = |a_x|_\infty$. $(u,v)$ and $||u||$ denote the standard $L^2$ scalar product and norm.

Let $v_j(t)$ be the semi-discrete approximation of $u(x_j,t)$. We discretize (8) in space by mixing the standard and extended SBP operators,

$$\frac{dv}{dt} = -\frac{1}{2}a\widetilde{D}\tilde{v} - \frac{1}{2}D(av) + \frac{1}{2}D(a)v. \tag{11}$$

To derive an energy estimate, we form the scalar product between $v$ and (11),

$$(v,v_t)_h = -\frac{1}{2}(v,a\widetilde{D}\tilde{v})_h - \frac{1}{2}(v,D(av))_h + \frac{1}{2}(v,D(a)v)_h.$$

We set $w = av$ in the first term on the right hand side. The SBP property (6) gives

$$(w,\widetilde{D}v)_h = -(Dw,v)_h - w_1(v_1 - \omega_1\beta^T\tilde{v}) + w_N(v_N - \omega_N\delta^T\tilde{v}).$$

Therefore,

$$(v,v_t)_h = \frac{1}{2}(v,D(a)v)_h + \frac{1}{2}\left[a_1v_1(v_1 - \omega_1\beta^T\tilde{v}) - a_Nv_N(v_N - \omega_N\delta^T\tilde{v})\right].$$

We can write

$$v_1(v_1 - \omega_1\beta^T\tilde{v}) = \left(v_1 - \frac{\omega_1}{2}\beta^T\tilde{v}\right)^2 - \frac{\omega_1^2}{4}(\beta^T\tilde{v})^2,$$

and the estimate for the semi-discrete problem becomes

$$\frac{1}{2}\frac{d||v||_h^2}{dt} = \frac{1}{2}(v, D(a)v)_h + \frac{a_1}{2}\left[\left(v_1 - \frac{\omega_1}{2}\beta^T\tilde{v}\right)^2 - \frac{\omega_1^2}{4}(\beta^T\tilde{v})^2\right] - \frac{a_N}{2}\left[\left(v_N - \frac{\omega_N}{2}\delta^T\tilde{v}\right)^2 - \frac{\omega_N^2}{4}(\delta^T\tilde{v})^2\right]. \quad (12)$$

The boundary data at $x = 0$ (inflow) can be enforced by choosing the ghost point value $v_0$ such that

$$v_1 - \frac{\omega_1}{2}\beta^T\tilde{v} = g(t). \quad (13)$$

At $x = 1$ (outflow), we choose the ghost point value $v_{N+1}$ such that

$$\delta^T\tilde{v} = 0, \quad (14)$$

which is an extrapolation formula. With the boundary conditions (13) and (14), we arrive the estimate

$$\frac{1}{2}\frac{d\|v\|_h^2}{dt} \le \frac{1}{2}\alpha_h\|v\|_h^2 + \frac{a_1}{2}g(t)^2,$$

where $\alpha_h = \max_j |D(a)_j|$. This estimate corresponds to (10) for (8).

If, for example, we use a diagonal norm SBP operator that is 6th order accurate in the interior of the domain, and 3rd order near the boundary, the solution can not be expected to be more than 4th order accurate. It is then reasonable to choose

$$\beta^T\tilde{v} = \kappa(v_0 - 4v_1 + 6v_2 - 4v_3 + v_4), \quad (15)$$

$$\delta^T\tilde{v} = \kappa(v_{N+1} - 4v_N + 6v_{N-1} - 4v_{N-2} + v_{N-3}), \quad (16)$$

where $\kappa$ is a tunable parameter. With this choice $\frac{1}{h}\beta^T\tilde{v} = \mathscr{O}(h^3)$, i.e., $\widetilde{D}v$ has a 3rd order truncation error near the boundary. Furthermore, (13) imposes the Dirichlet boundary condition to 4th order accuracy. Inserting (15) and (16) into (13) and (14), respectively, lead to the boundary conditions

$$v_0 = \frac{2(v_1 - g(t))}{\kappa\omega_1} + 4v_1 - 6v_2 + 4v_3 - v_4,$$
$$v_{N+1} = 4v_N - 6v_{N-1} + 4v_{N-2} - v_{N-3}.$$

*Remark 1.* In this simple example the ghost point value $v_{N+1}$ is only used to set $\delta^T\tilde{v} = 0$. We could therefore have defined $\widetilde{D}$ without the term $\mathbf{e}_N(\delta^T\tilde{v})$, i.e., we obtain the standard SBP procedure where no boundary condition is explicitly needed at outflow boundaries. Furthermore, if the ghost point $v_0$ is eliminated from (11) for $j = 1$, it turns out that the term $-a\frac{v_1 - g}{h\omega_1}$ appears. Hence, for this simple semi-discrete problem the proposed technique is equivalent with an SAT method.

## 3 Elastic-acoustic coupled problem

We consider a one dimensional domain of length $2L$, $-L \le x \le L$, with an elastic-acoustic interface at $x = 0$. The domain to the left, $-L \le x \le 0$, is a solid described by the wave equation

$$\rho_e w_{tt} = (\mu w_x)_x + g, \quad t > 0 \quad -L \le x \le 0 \tag{17}$$

where $w$ is the displacement, $\rho_e(x)$ is the density of the solid, $\mu(x)$ its shear modulus, and $g = g(x,t)$ is a given forcing function. The domain to the right, $0 \le x \le L$, is acoustic and described by the linearized and symmetrized Euler equations,

$$\mathbf{q}_t + A(x)\mathbf{q}_x = E(x)\mathbf{q} + \mathbf{f}, \tag{18}$$

where $\mathbf{q} = (s,\ u,\ \tilde{p})$, with

$$s = \frac{1}{\hat{\rho}\hat{c}}p - \frac{\hat{c}}{\hat{\rho}}\rho \qquad \tilde{p} = \frac{1}{\hat{\rho}\hat{c}}p,$$

and where $\rho$, $u$, and $p$ are the density, velocity, and pressure perturbations in the air. The hat variables denote a given, steady, background field $(\hat{\rho}(x),\ \hat{u}(x),\ \hat{c}(x))$, where the background pressure is given by $\hat{p} = \hat{c}^2\hat{\rho}/\gamma$. Here $\gamma$ is a constant, usually taken to be 1.4 in air. The matrices are given by

$$A = \begin{pmatrix} \hat{u} & 0 & 0 \\ 0 & \hat{u} & \hat{c} \\ 0 & \hat{c} & \hat{u} \end{pmatrix} \qquad E = \begin{pmatrix} \hat{u}_x - 3\frac{\hat{u}}{\hat{c}}\hat{c}_x + \frac{\hat{u}}{\hat{p}}\hat{p}_x & \frac{\gamma-1}{\hat{\rho}\hat{c}}\hat{p}_x + 2\hat{c}_x & (\gamma-1)\hat{u}_x + 2\frac{\hat{u}}{\hat{c}}\hat{c}_x \\ \frac{1}{\hat{\rho}\hat{c}}\hat{p}_x & \hat{u}_x & \frac{\gamma-1}{\hat{\rho}\hat{c}}\hat{p}_x - \hat{c}_x \\ 0 & \frac{1}{\hat{\rho}\hat{c}}\hat{p}_x & \gamma\hat{u}_x - \frac{\hat{u}}{\hat{c}}\hat{c}_x + \frac{\hat{u}}{\hat{p}}\hat{p}_x \end{pmatrix}$$

where we note that $A(x)$ is symmetric. The function $\mathbf{f} = \mathbf{f}(x,t)$ is a given forcing function. At the interface, the background velocity is assumed to vanish, $\hat{u}(0) = 0$.

A grid with grid spacing $h$, $x_j = jh$ discretizes the domain. The interface is located at $x_0 = 0$. Here, $x_{-1}$ is a ghost point for the acoustic domain, and $x_1$ is ghost point for the elastic domain. Fig. 1 shows the grid points of the acoustic (blue) and elastic (red) domains near the interface. Similarly to the scalar problem in Sec. 2,

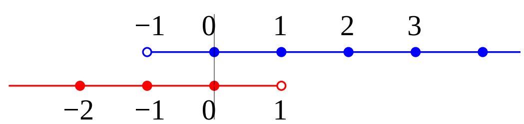


**Fig. 1** Grid $x_j = jh$ near the interface at $j = 0$. The elastic domain (red) uses a ghost point at $j = 1$. The acoustic domain (blue) has a ghost point at $j = -1$.

the acoustic equations are discretized in space by

$$\frac{d}{dt}\mathbf{q}_j = -\frac{1}{2}A_j\widetilde{D}\mathbf{q}_j - \frac{1}{2\hat{\rho}_j}D(\hat{\rho}A\mathbf{q})_j + F_j\mathbf{q}_j + \mathbf{f}_j \tag{19}$$

for $j = 0,1,\ldots,N$. The matrix $F = E + \frac{1}{\hat{\rho}}(\hat{\rho}A)_x$. The density weighting in the splitting is introduced to ensure that the scaling of the boundary terms at the interface matches the scaling of the boundary term from the wave equation in the elastic domain. Denote the SBP scalar product on the acoustic domain by $(\mathbf{u},\mathbf{v})_{h+} = h\sum_{j=0}^{N}\omega_j^+\mathbf{u}_j^T\mathbf{v}_j$, where $\omega_j^+$ are the SBP norm weights. The spatial discretization satisfies the estimate (where we set $\mathbf{f} = \mathbf{0}$),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\hat{\rho}\mathbf{q},\mathbf{q})_{h+} &= (\mathbf{q},\hat{\rho}\mathbf{q}_t)_{h+} = -\frac{1}{2}(\mathbf{q},\hat{\rho}A\widetilde{D}\mathbf{q})_{h+} - \frac{1}{2}(\mathbf{q},D(\hat{\rho}A\mathbf{q}))_{h+} + (\mathbf{q},\hat{\rho}F\mathbf{q})_{h+}\\ &= \frac{1}{2}\mathbf{q}_0^T\hat{\rho}_0A_0(\mathbf{q}_0-\beta\mathbf{q}_0) - \frac{1}{2}\mathbf{q}_N^T\hat{\rho}_NA_N(\mathbf{q}_N-\delta\mathbf{q}_N) + (\mathbf{q},\hat{\rho}F\mathbf{q})_{h+}.\end{aligned} \tag{20}$$

This can be seen by straightforward generalization of the scalar identity (6) and by using the symmetry of $A$. The assumption $\hat{u}(0) = 0$ implies that the boundary term at $x = x_0$ can be written

$$\frac{1}{2}\mathbf{q}_0^T\hat{\rho}_0A_0(\mathbf{q}_0-\beta\mathbf{q}_0) = (u_0-\beta u_0)(p_0-\beta p_0) - (\beta u_0)(\beta p_0)$$

Here we use the notation $\beta u_0 = \sum_{k=-1}^{r-1}\beta_{k+1}u_k$ for the operator (4) and similarly for $\delta u_N$. When applied to vectors, $\beta\mathbf{q}$ is defined component wise, $\beta\mathbf{q} = (\beta s,\ \beta u,\ \beta\tilde{p})$. In order to advance in time with the same method in the acoustic and elastic domains, we rewrite (17) as a system of two equations with first derivatives in time. After discretizing in space we obtain,

$$\frac{dv_j}{dt} = G(\mu,w)_j + g_j \qquad \frac{dw_j}{dt} = v_j, \tag{21}$$

for $j = -N,\ldots,0$. The spatial discretization $G(\mu,w)$ is the SBP operator approximating $(\mu u_x)_x$, developed in [2]. It satisfies, in the SBP scalar product $(v,w)_{h-}$,

$$(v,G(\mu,w))_{h-} = -(Dv,\mu Dw)_{h-} - (v,Pw)_{h-} - v_{-N}\mu_{-N}Sw_{-N} + v_0\mu_0Sw_0, \tag{22}$$

where $P$ is a positive semi-definite operator that is small and $Sw_0$ is a high order approximation of $w_x(x_0)$ using the stencil $w_{-m},\ldots,w_1$, for some stencil width $m+1$.

The energy norm, $N_E$, of the solution over both domains satisfies

$$\begin{aligned}\frac{1}{2}\frac{dN_E}{dt} &= \frac{1}{2}\frac{d}{dt}\left((w_t,\rho_e w_t)_{h-} + (Dw,\mu Dw)_{h-} + (w,Pw)_{h-} + (\mathbf{q},\hat{\rho}\mathbf{q})_{h+}\right)\\ &= (v,\rho_e v_t)_{h-} + (Dv,\mu Dw)_{h-} + (v,Pw)_{h-} + (\mathbf{q},\hat{\rho}\mathbf{q}_t)_{h+}\\ &= v_0\mu_0Sw_0 + \frac{1}{2}\mathbf{q}_0^T\hat{\rho}_0A_0(\mathbf{q}_0-\beta\mathbf{q}_0) + (\mathbf{q},\hat{\rho}F\mathbf{q})_{h+} + B_2,\end{aligned} \tag{23}$$

which can be seen by combining (20) and (22). Here, $B_2$ denotes boundary terms at the boundaries at $x = \pm L$, and we have set $g = 0$ in (21). The interface conditions will be stable if the boundary terms at the interface do not contriubte to any norm increase, i.e., if

$$v_0 \mu_0 S w_0 + (u_0 - \beta u_0)(p_0 - \beta p_0) - (\beta u_0)(\beta p_0) = 0. \tag{24}$$

We enforce the discrete interface condition (24), by setting

$$\beta p_0 = 0 \tag{25}$$

$$\mu_0 S w_0 = -(p_0 - \beta p_0) \tag{26}$$

$$v_0 = u_0 - \beta u_0. \tag{27}$$

Here (25) determines $p_{-1}$, (26) determines $w_1$, and (27) determines $u_{-1}$. This means that stress and velocity are required to be continuous across the interface.

Alternatively, the approximation for the acoustic equations can be done without use of ghost points. In that case, the operator $\widetilde{D}$ in (19) is replaced by $D$, and the extra operator $\beta = 0$, which give

$$v_0 \mu_0 S w_0 = -p_0 \qquad \text{and} \qquad v_0 = u_0.$$

These two conditions are used to determine $w_1$ and $u_0$, respectively. Here, the acoustic velocity is set by direct injection. This is equivalent with the projection method, and hence, also leads to a stable method.

## 4 Numerical experiments

The semi-discrete acoustic-elastic problem (19), (21) with interface conditions (25)–(27) was solved in time by the fourth order accurate Runge-Kutta method. The SBP first derivative operator $D$ of order six interior and three on the boundary was used in (19). The SBP operator $G(\mu, w)$ of fourth order accuracy in the interior and second order on the boundary, developed in [2], was used in (21).

The domain is $-L \le x \le L$, with $L = 1000$. The grid near the interface is as outlined in Fig. 1. The acoustic background state is

$$(\hat{\rho},\ \hat{u},\ \hat{c}) = (1 + \cos(\omega_m x + \phi_1)/5,\ 10\sin(\omega_m x),\ 340 - 30\sin(\omega_m x + \phi_2))$$

and elastic material is

$$\rho_e = 2600 + 150\cos(\omega x + \phi_2) \qquad \mu = \rho_e c^2, \quad c = 1000 + 400\sin(\omega x + \phi_1).$$

These material properties have sizes that are realistic for a seismo-acoustic computation. The manufactured solution for the elastic domain is

$$w(x,t) = \sin(210\omega t - \phi_1)\cos(-2\omega(x-200t)-\phi_1)$$

and the acoustic manufactured solution is

$$\begin{pmatrix}\rho\\ u\\ p\end{pmatrix} = \begin{pmatrix}\cos(\omega x)\sin(\omega(x-150t))/20\\ \sin(\omega x+\phi_1)\cos(420\omega t)\\ 200\sin(\omega x)\sin(170\omega t+\phi_2)\end{pmatrix}$$

The parameters have values $\omega = 0.023$, $\omega_m = 0.021$, $\phi_1 = 0.17$, and $\phi_2 = 0.08$. The source functions $\mathbf{f}(x,t)$ and $g(x,t)$ in (18) and (17) are determined to yield the manufactured solutions as solutions. Forcing functions are also inserted into the interface conditions. These are needed to enforce the jump in the manufactured solutions across the interface. Finally, all time dependent boundary forcing and interface forcing functions are modified, as described in [1], when imposed during the Runge-Kutta stages, to make the time discretization achieve full fourth order. The convergence under grid refinement is shown in Fig. 2. Fourth order convergence is observed in all variables except in the acoustic density, which converges somewhere between third and fourth order. This is probably due to the interface being a characteristic boundary for the acoustic equations, since recovery of fourth order convergence from the third order truncation error on the boundary is not guaranteed at such boundaries.

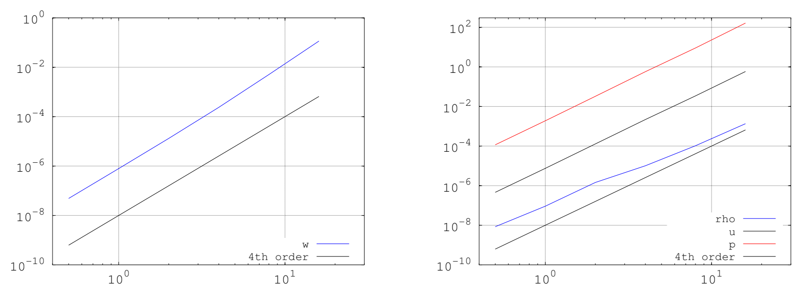


**Fig. 2** Maximum norm errors of the manufactured solution at $t = 1$ vs. grid spacing. Left subplot shows the error in the elastic variable $w$, the right subplot shows errors in the acoustic density (blue), velocity (black), and pressure (red). Thin black lines show 4th order reference.

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