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Sequential Detection of Fission Processes for Harbor Defense

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***Abstract*—With the large increase in terrorist activities throughout the world, the timely and accurate detection of special nuclear material (*SNM*) has become an extremely high priority for many countries concerned with national security. The detection of radionuclide contraband based on their γ-ray emissions has been attacked vigorously with some interesting and feasible results; however, the fission process of *SNM* has not received as much attention due to its inherent complexity and required predictive nature. In this paper, on-line, sequential Bayesian detection and estimation (parameter) techniques to rapidly and reliably detect unknown fissioning sources with high statistical confidence are developed.**

I. INTRODUCTION

The detection of special nuclear material (*SNM*) is becoming more and more of a concern as terrorist activities throughout the world have increased dramatically. Sequential detection techniques have been proven to provide a highly reliable detection process with high confidence minimizing false alarms capable of making decisions in a timely manner [1], [2]. We develop a sequential Bayesian methodology to cast the basic decision problem into this framework. Here we exploit the fissioning nature of the threat to obtain unique signatures that can be utilized for *SNM* detection.

Previous work in the *SNM* detection area applying Bayesian techniques have been applied to the radiation detection problem [3]-[6] where a sequential algorithm coupled to a parameter estimation scheme enables the detection of threat materials based on the gamma-rays emitted by the targeted radionuclide [7]-[19]. Here we concentrate on neutron emissions measured directly by a neutron multiplicity counter that counts the arrivals.

The implementation of a sequential Bayesian detector for this problem requires a-priori information about the unknown source(s) that is being targeted for detection much the same as a sequential radiation detection scheme [7]-[14]. Therefore, a sophisticated estimation scheme must be *embedded* in the detector to provide updated parameter estimates of the source in order to be successful.

The development of a Bayesian particle filter (*PF*) or equivalently sequential Markov Chain Monte Carlo (*MCMC*) processor for a fissioning process is quite a challenging problem due to its underlying stochastic nature. This effort is based on the fundamental theory and modeling developed by Prasad and Snyderman [20], [21] and summarized in Walston [22]. The fission process has a rich history of development and its comprehension has evolved through a wide variety of research [23], [24]. Even though the underlying theory is complex, it can still be captured by stochastic modeling. Here a Bayesian approach to the problem is developed based on the underlying time-interval probability distribution [20].

In Sec. II, we define the Bayesian problem by discussing the underlying fission physics leading to the inter-arrival time or equivalently the time-interval distribution [20]. Once established, the sequential Bayesian detection framework is developed incorporating the source information into the decision function for both known and unknown parameters in Sec. III. Next the generalized case is developed for unknown source parameters that must be estimated (on-line) from the incoming neutron arrivals. Once the structure is established for this case, Bayesian parameter estimation schemes are developed in Sec. IV leading to the sequential particle filter. The general structure, motivated by the fission problem, is then implemented in Sec. V and applied to simulated arrival data.

II. PHYSICS-BASED MODELS FOR DETECTION

In this section we discuss the development of a physics-based fission model capturing the joint distribution of the overall fission processes and their underlying statistics. As mentioned above, this development is based on the fundamental theory and modeling developed in Refr. [20]. We start with the basic neutron physics and progress to the full propagation leading to the desired probability distribution.

A. Fission-Based Processing Model

A neutron arriving at a detector at energy level α and arrival time t_m can be characterized as a single impulse $\epsilon_m \delta(t - t_m)$. A train of neutrons is defined as a set of arrivals that do *not* overlap in time. *Inter-arrival times* are defined as $\tau_m := t_m - t_{m-1}$ for $m = 0, 1, \cdots, M$ and the complete set of inter-arrivals by $T_M := \{\tau_0, \tau_1, \cdots, \tau_M\}$. These arrivals are measured by a neutron multiplicity counter [25] that is basically a sophisticated neutron detector that evaluates the inter-arrival time probability distribution of neutrons emitted spontaneously by fissionable materials. Recall that the radioactive decay of each unstable nucleus produces multiple neutrons that can interact with other nuclei exciting them to energy levels enabling them to split into smaller unequal fission fragments which are also unstable and decay even further (emitting neutrons) toward stable nuclei. The detection of these neutrons, which can pass through heavy shielding, provides a methodology to detect special nuclear material. The key issue with threat materials is that the number of neutrons released are produced by a single decay defining its *multiplication*.

These "correlated" neutrons offer a unique *SNM* signature that indicates both its multiplication and the mass of spontaneous fission isotopes. Thus, the *neutron multiplicity counter* or *neutron detector* is a stochastic measurement system creating an estimated time-interval probability distribution that is used in the neutron detection process to alarm on *SNM*. Thus, the essential ingredient of these measurements is its underlying time-interval probability distribution.

A simplistic model of the multiplication process consists of a fission chain generated by a spontaneous fission under the assumption that the source is characterized by an inhomogeneous Poisson process with a varying fission rate [26], [27]. During a single fission chain ν-neutrons are emitted with probability P_ν. These emissions are slowed in a moderator and diffuse exponentially in time as thermal neutrons. Then the probability that n of the ν neutrons are absorbed in the detector and converted to electrical pulses is

$$\Pr[\nu; T_M] = P_\nu\binom{\nu}{n}\epsilon^n(1-\epsilon)^{\nu-n} \tag{1}$$

where ϵ is the probability of neutron detection (detection efficiency). Mathematically, we can represent the N_ν-neutron burst sequence emitted by

$$\eta(\tau_m;\Theta_S) = \sum_{i=1}^{N_\nu}\alpha_i\delta(\tau-\tau_m(\Theta_S)); m=0,\cdots,M; \tau_m < T_M \tag{2}$$

for α the i-th energy of the neutron at the m-th inter-arrival in the M-th time interval. Typically, we ignore the neutron energy and concentrate on the inter-arrival, since the source information is contained in $\tau_m(\Theta_S)$.

For spontaneous fissions, the quantity, k_{eff}, is the average number of neutrons from one fission that initiates another [3]. Any remaining neutrons are absorbed or escape. The value of k_{eff} specifies how a chain will proceed. For instance, the $k_{\text{eff}} = 1$ (critical mass) leads to a fission level that is constant and is typical to power plant operation, while $k_{\text{eff}} > 1$ (super-critical mass) for an event implies that there may be k_{eff}-events to follow which is typical in weapons applications. For the latter, the number of fission reactions *increase* exponentially. We will use Eq. 2 in developing the subsequent fission detection schemes to follow.

B. Inter-arrival Distribution

Theoretically, the conditional distribution of inter-arrival times τ conditioned on a set of source parameters Θ (following [20], [21]) is given by

$$\Pr[\tau|\Theta] = \underbrace{R_1 r_0 \mathrm{n}_0}_{Time\ between\ chain\ initiations} + \underbrace{\frac{F_S}{R_1}\sum_{n=2}^{\infty}\mathrm{e}_n(\epsilon)\left(\sum_{k=1}^{n-1}ke^{-k\lambda\tau}\right)\lambda b_0(\tau)}_{Time\ between\ neutrons\ in\ same\ chain} \tag{3}$$

where

R_1 is the count rate;
r_0 is the probability that NO neutrons are detected within the time-interval τ;
n_0 is the probability of zero counts in time interval τ;
τ is the time interval or inter-arrival time;
F_S is the fission rate;
$\mathrm{e}_n(\epsilon)$ is the probability of detecting *n* neutrons from the same fission chain;
λ is the inverse of the diffusion time scale; and
$b_0(\tau)$ is the probability of NO counts in time interval τ.

Embedded in Eq. 3 is a set of various relations that capture the time-interval probability:

$$Fs = \frac{N_A}{A}\frac{\ln 2\ t_{1/2}}{t_{1/2}\ t_{1/2}^{SF}}m_{\mathrm{S}}$$

$$R_1 = \epsilon\ q\ \mathcal{M}\ \overline{\nu}_S\ F_S$$

for

m_{S} mass of the source;
ϵ is the detection efficiency;
p is the probability that a neutron induces a fission;
q is the escape probability ($q = 1-p$);
$\mathcal{M}$ is the system multiplication;
$\overline{\nu}_S$ is the average neutron count from a spontaneous fission;
$\overline{\nu}_I$ is the average neutron count from an induced fission;
N_A is Avogadro's number;
A is the atomic weight; and
$t_{1/2}$ is the half-life.

with *multiplication* given by

$$\mathcal{M} = \frac{1}{1-p\overline{\nu}_I} = \frac{1}{1-k_{\text{eff}}}$$

for k_{eff} the *effective multiplication* and *detection efficiency* approximated by

$$\epsilon = \frac{a}{b\mathcal{M}_e^2 + c\mathcal{M}_e}; \quad 0 \le \epsilon \le 0.04; \quad a,b,c \text{ fit parameters}$$

where $\mathcal{M}_e = q \times \mathcal{M}$. The *probability* of detecting n-neutrons of the ν emitted with probability P_ν is given by

$$\mathrm{e}_n(\epsilon) = \sum_{\nu=n}^{\infty}P_\nu\binom{\nu}{n}\epsilon^n(1-\epsilon)^{\nu-n}$$

The following probabilities complete the distribution

$$r_0 = \frac{F_S}{R_1}\sum_{n=1}^{\infty}\mathrm{e}_n(\epsilon)\Big(\sum_{k=0}^{n-1}e^{-k\lambda\tau}\Big)$$

and

$$b_0(\tau) = \exp\Big[-F_S\int_0^\tau\frac{1}{1-e^{-\lambda t}}\Big(1-\sum_{\nu=0}^{\infty}P_\nu(1-\epsilon(1-e^{-\lambda t}))^\nu\Big)dt\Big]$$

$$\mathrm{n}_0(\tau) = r_0(\tau) \times b_0(\tau)$$

Since the objective is to "decide" whether or not a fissioning source is present, we require a-priori knowledge of the source parameters: mass, multiplication, detection efficiency and diffusion time scale parameters. Notationally, we define the source parameters as $m_{\mathrm{S}}, k_{\mathrm{eff}}, \epsilon$ and λ respectively and note their intimate relations in the overall probability distribution function.

III. SEQUENTIAL DETECTION FOR FISSION PROCESSES

In order to develop a sequential processor [28], [29], we must test the binary hypothesis that the measured inter-arrival times have evolved from a fissioning *SNM* threat. The basic decision problem is simply stated as:

GIVEN a set of uncertain neutron multiplicity detector inter-arrival measurements $\{\tau_m\}; m = 0, 1, \cdots, M$ from an unknown source, DECIDE whether or not the source is a threat (*SNM*). If so, "extract" its characteristic parameters, Θ to "classify" its type.

We are to test the hypothesis that the set of measured neutron inter-arrivals T_M have evolved from a threat or non-threat source. Therefore, we specify the hypothesis test by

$$\begin{aligned} \mathcal{H}_0 :\ & T_M = \mathcal{T}_B\left(\underline{\mathrm{m}}; \Theta_b\right) + \mathcal{T}_V\left(\underline{\mathrm{m}}\right) && \text{[NON-THREAT]} \\ \mathcal{H}_1 : & T_M = \mathcal{T}_S\left(\underline{\mathrm{m}}; \Theta_s\right) + \mathcal{T}_B\left(\underline{\mathrm{m}}; \Theta_b\right) + \mathcal{T}_V\left(\underline{\mathrm{m}}\right) && \text{[THREAT]} \end{aligned} \tag{4}$$

where $\mathcal{T}_S$ is the unknown source inter-arrivals with parameters Θ_s, $\mathcal{T}_B$ is the background inter-arrivals (cosmic rays, etc.) with parameters Θ_b, $\mathcal{T}_V$ is the zero-mean, Gaussian measurement (instrumentation) inter-arrival noise, $T_M := \{\tau_0, \tau_1, \cdots, \tau_M\}$ and $\underline{\mathrm{m}} := 0, 1, \cdots, M$.

The fundamental approach of classical detection theory to solving this binary decision problem is to apply the Neyman-Pearson criterion of maximizing the detection probability for a specified false alarm rate [28] with the parameters Θ *known*. The result leads to a *likelihood ratio* decision function defined by [28], [29]

$$\mathcal{L}(T_M; \Theta) := \frac{\Pr[T_M|\Theta; \mathcal{H}_1]}{\Pr[T_M|\Theta; \mathcal{H}_0]} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \mathcal{T} \tag{5}$$

with threshold $\mathcal{T}$. This expression implies a "batch" decision, that is, we gather the M inter-arrivals T_M, calculate the likelihood (Eq. 5) over the entire batch of data and compare it to the threshold $\mathcal{T}$ to make the decision.

A. Sequential Processor

An alternative to the batch approach is the sequential method which can be developed by expanding the likelihood ratio for each inter-arrival to obtain the recursion or equivalently *sequential likelihood ratio* for the m-th inter-arrival follows as

$$\mathcal{L}(T_m; \Theta) = \mathcal{L}(T_{m-1}; \Theta) \times \frac{\Pr[\tau_m|T_{m-1}, \Theta; \mathcal{H}_1]}{\Pr[\tau_m|T_{m-1}, \Theta; \mathcal{H}_0]}; m = 0, \cdots, M \tag{6}$$

with $\Pr[\tau_0|T_{-1}, \Theta; \mathcal{H}_\ell] = \Pr[\tau_0|\Theta; \mathcal{H}_\ell]$, the *prior* under each hypothesis.

The Wald *sequential probability-ratio test* is [1], [2]

$$\begin{array}{lll} \mathcal{L}(T_m; \Theta) & > \mathcal{T}_1(m) & \text{Accept} \quad \mathcal{H}_1 \\ \mathcal{T}_0(m) \quad \leq \mathcal{L}(T_m; \Theta) & \leq \mathcal{T}_1(m) & \text{Continue} \\ \mathcal{L}(T_m; \Theta) & < \mathcal{T}_0(m) & \text{Accept} \quad \mathcal{H}_0 \end{array} \tag{7}$$

where the thresholds are specified in terms of the false alarm (P_{FA}) and miss (P_M) probabilities as

$$\mathcal{T}_0(m) = \frac{\mathrm{P}_M(m)}{1 - \mathrm{P}_{FA}(m)} \qquad \mathcal{T}_1(m) = \frac{1 - \mathrm{P}_M(m)}{\mathrm{P}_{FA}(m)} \tag{8}$$

These thresholds are determined from a receiver operating characteristic (*ROC*) curve (detection versus false alarm probabilities) obtained by simulation or a controlled experiment to calculate the decision function. That is, an operating point is selected from the *ROC* corresponding to specific detection (or equivalently miss) and false-alarm probabilities specifying the required thresholds which are calculated according to Eq. 8 for each parameter update.

A reasonable approach to this problem of making a reliable decision with high confidence in a timely manner is to develop a sequential detection processor. At each neutron arrival (at τ_m), we *sequentially update* the decision function and compare it to the thresholds to perform the detection—"neutron-by-neutron". Here as each neutron is monitored producing the inter-arrival sequence, the processor takes each inter-arrival measurement and attempts to "decide" whether or not it evolves from a threat or non-threat. For each inter-arrival, the decision function is "sequentially" updated and compared to the detection thresholds obtained from the *ROC* curve operating point enabling a rapid decision. Once the threshold is crossed, the decision (threat or non-threat) is made and the arrival is processed; however, if not enough data is available to make the decision, then another measurement is obtained.

For our problem, we typically have information about the background, disturbance and noise parameters, but we rarely have the source information. Therefore, we still can make a decision, but require estimates of the unknown parameters, that is, $\hat{\Theta} \rightarrow \Theta$. In this case, we must construct a *composite* or *generalized* likelihood-ratio test (*GLRT*).

Therefore from the batch likelihood decision function, we can consider the two cases of the *GLRT*: (i) parameters are *random* or (ii) parameters are *deterministic* but *unknown*.

Here we assume that Θ is deterministic but unknown. Therefore, the approach is to estimate the unknown parameter vector $\hat{\Theta} \rightarrow \Theta$ under each hypothesis and proceed with the simple testing. A *maximum likelihood estimate* $\hat{\Theta}_{ML}$, can be used to create the *GLRT* such that

$$\mathcal{L}(T_M;\Theta) = \frac{\max\limits_{\Theta_1} \quad \Pr[T_M|\Theta_1;\mathcal{H}_1]}{\max\limits_{\Theta_0} \quad \Pr[T_M|\Theta_0;\mathcal{H}_0]} \tag{9}$$

This is the approach we employ *initially*. The batch solution for the *GLRT* can also be extended to the sequential case as before giving the solution by simply replacing $\hat{\Theta}_{ML} \rightarrow \Theta$, that is,

$$\begin{aligned}\mathcal{L}(T_m;\hat{\Theta}) &= \mathcal{L}(T_{m-1};\hat{\Theta}) \times \frac{\Pr[\tau_m|T_{m-1},\hat{\Theta}_1;\mathcal{H}_1]}{\Pr[\tau_m|T_{m-1},\hat{\Theta}_0;\mathcal{H}_0]}; \\ & m = 0,1,\cdots,M\end{aligned} \tag{10}$$

Anticipating Gaussian models (exponential family [36]) for our unknown parameters, we develop the logarithmic form of the sequential likelihood decision function. Simply taking the natural logarithm of Eq. 10, that is, $\Lambda(T_M;\Theta) := \ln \mathcal{L}(T_M;\Theta)$ we obtain the *log-likelihood* sequential decision function as

$$\begin{aligned}&\Lambda(T_m;\hat{\Theta}) = \Lambda(T_{m-1};\hat{\Theta}) + \\ &\ln \Pr[\tau_m|T_{m-1},\hat{\Theta}_1;\mathcal{H}_1] - \ln \Pr[\tau_m|T_{m-1},\hat{\Theta}_0;\mathcal{H}_0]\end{aligned} \tag{11}$$

Using these formulations, we develop the detection algorithm for our problem next. We should note that we only consider the "threat detection problem" in this paper.

B. Sequential Detection for Fission Processes

Here we start with the results of the previous section and incorporate the physics of the fission process. For fission detection, we start with the simple neutron model of Eq. 2 at inter-arrival time τ_m leading to the subsequent (sequential) hypothesis test:

$$\begin{aligned}\mathcal{H}_0 :\ \tau_m &= \mathcal{T}_B\Big(m;\Theta_b\Big) + \mathcal{T}_V\Big(m\Big) \quad \text{[NON-THREAT]} \\ \mathcal{H}_1 : \tau_m &= \mathcal{T}_S\Big(m;\Theta_s\Big) + \mathcal{T}_B\Big(m;\Theta_b\Big) + \mathcal{T}_V\Big(m\Big) \text{[THREAT]}\end{aligned} \tag{12}$$

The sequential detection solution (as before) for this problem with unknown source parameters follows directly from the *GLRT* results of Eq. 10. To implement the processor, we must first determine the required conditional probabilities in order to specify the decision function, that is,

$$\begin{aligned}\Pr[\tau_m|T_{m-1},\hat{\Theta};\mathcal{H}_1] &= \Pr[\mathcal{T}_S(m;\Theta_S)|T_{m-1},\hat{\Theta}_S;\mathcal{H}_1] + \\ &\Pr[\mathcal{T}_B(m;\Theta_b)|T_{m-1},\Theta_b;\mathcal{H}_1] + \\ &\Pr[\mathcal{T}_V(m)]\end{aligned} \tag{13}$$

and under the null hypothesis

$$\Pr[\tau_m|T_{m-1},\Theta;\mathcal{H}_0] = \Pr[\mathcal{T}_B(m;\Theta_b)|T_{m-1},\Theta_b;\mathcal{H}_0] + \Pr[\mathcal{T}_V(m)] \tag{14}$$

where the Gaussian inter-arrival noise is distributed as $\mathcal{T}_V \sim \mathcal{N}(0,\sigma_{vv}^2)$ and the known background disturbances [15] are ignored (at this point) while the inter-arrival distribution is specified (instantaneously at τ_m) by Eq. 3 with $\tau_m \rightarrow \tau$ to give:

$$\begin{aligned}&\Pr[\mathcal{T}_S(m;\Theta_S)|T_{m-1},\hat{\Theta}_S;\mathcal{H}_1] = R_1 r_0 \mathrm{n}_0 + \\ &\frac{F_S}{R_1}\sum_{n=2}^{\infty} \mathrm{e}_n(\epsilon)\left(\sum_{k=1}^{n-1} k e^{-k\lambda\tau_m}\right)\lambda b_0(\tau_m)\end{aligned} \tag{15}$$

$$\Pr[\mathcal{T}_V(m)] = \frac{1}{\sqrt{2\pi\sigma_{vv}^2}} \exp\Big\{-\frac{1}{2}\frac{\mathcal{T}_V^2(m)}{\sigma_{vv}^2}\Big\} \tag{16}$$

Therefore, the log-likelihood ratio becomes (simply)

$$\begin{aligned}&\Lambda(T_m;\Theta) = \Lambda(T_{m-1};\Theta) + \\ &\ln\Big[R_1 r_0 \mathrm{n}_0 + \frac{F_S}{R_1}\sum_{n=2}^{\infty} \mathrm{e}_n(\epsilon)\left(\sum_{k=1}^{n-1} k e^{-k\lambda\tau_m}\right)\lambda b_0(\tau_m) \\ &+\frac{1}{\sqrt{2\pi\sigma_{vv}^2}} \exp\Big\{-\frac{1}{2}\frac{\mathcal{T}_V^2(m)}{\sigma_{vv}^2}\Big\}\Big] \\ &-\ln\Big(\frac{1}{\sqrt{2\pi\sigma_{vv}^2}}\Big) + \frac{1}{2}\frac{\mathcal{T}_V^2(m)}{\sigma_{vv}^2}\end{aligned} \tag{17}$$

We can also extend the problem to the *random* case. Suppose we assume that each of the independent source parameters are governed by a *random walk/random constant* model, that is, the parameters are assumed piecewise constant and subjected to zero-mean, Gaussian, random uncertainties, w_Θ, with covariance, $R_{w_\Theta w_\Theta}$ driving the process [32], then in this case we have that the parameters, $\Theta \sim \mathcal{N}(\Theta_0, P_0 + R_{w_\Theta w_\Theta})$, where Θ_0 is the initial mean with P_0 its corresponding covariance. A more detailed discussion follows subsequently when we discuss the underlying parameter estimation problem in Sec. IV (see Eq. 30). Thus, the corresponding multivariate prior distribution is given by

$$\begin{aligned}&\Pr[\Theta(\tau_m)] = (2\pi)^{-N_\Theta/2}|P_0 + R_{w_\Theta w_\Theta}|^{-1/2} \times \\ &\exp\Big\{-1/2\ (\Theta-\Theta_0)'\Big(P_0 + R_{w_\Theta w_\Theta}\Big)^{-1}(\Theta-\Theta_0)\Big\}\end{aligned} \tag{18}$$

and the log-likelihood ratio is

$$\Lambda(T_m;\Theta) = \Lambda(T_{m-1};\Theta) + \ln \Pr[\tau_m, \Theta(\tau_m)|T_{m-1};\mathcal{H}_1] - \ln \Pr[\tau_m|T_{m-1},\Theta;\mathcal{H}_0] \tag{19}$$

where the second term can be expanded further by applying Bayes' rule to give

$$\begin{aligned}\Pr[\tau_m, \Theta(\tau_m)|T_{m-1};\mathcal{H}_1] &= \Pr[\tau_m|T_{m-1},\Theta(\tau_m);\mathcal{H}_1] \\ &\times \Pr[\Theta(\tau_m)|T_{m-1};\mathcal{H}_1]\end{aligned} \tag{20}$$

Substituting this expression for the source distribution of Eq. 13 gives

$$\begin{aligned}\Pr[\mathcal{T}_S(m;\Theta_S)|T_{m-1},\hat{\Theta}_S;\mathcal{H}_1] &\longrightarrow \Pr[\tau_m|T_{m-1},\Theta(\tau_m);\mathcal{H}_1] \\ &\times \Pr[\Theta(\tau_m)|T_{m-1};\mathcal{H}_1]\end{aligned} \tag{21}$$

With this in mind, the sequential log-likelihood can be calculated directly by substituting the prescribed distributions into Eq. 19 to give

$$\begin{aligned}&\Lambda(T_m;\Theta) = \Lambda(T_{m-1};\Theta) + \ln\Big(R_1 r_0 \mathrm{n}_0 + \\ &\frac{F_S}{R_1}\sum_{n=2}^{\infty}\mathrm{e}_n(\epsilon)\left(\sum_{k=1}^{n-1} k e^{-k\lambda\tau_m}\right)\lambda b_0(\tau_m) \\ &+(2\pi)^{-N_\Theta/2}|P_0 + R_{w_\Theta w_\Theta}|^{-1/2} \\ \times\ &\exp\Big\{-1/2\ (\Theta-\Theta_0)'\Big(P_0 + R_{w_\Theta w_\Theta}\Big)^{-1}(\Theta-\Theta_0)\Big\} \\ +\ &\frac{1}{\sqrt{2\pi\sigma_{vv}^2}}\exp\Big\{-\frac{1}{2}\frac{\mathcal{T}_V^2(m)}{\sigma_{vv}^2}\Big\}\Big) + \frac{1}{2}\ln\Big(2\pi\sigma_{vv}^2\Big) \\ &-\frac{1}{2}\frac{\mathcal{T}_V^2(m)}{\sigma_{vv}^2}\end{aligned} \tag{22}$$

This completes the development of the sequential Bayesian detection approach for fission processes. Next we must consider the parameter estimation problem in more detail.

IV. BAYESIAN PARAMETER ESTIMATION

In order to implement the *GLRT* of the previous section, we must estimate the unknown parameters Θ at each arrival. We first develop the batch scheme and then its sequential version similar to the sequential Bayesian detector of the previous section. Here we develop the Bayesian parameter estimator that can be applied to the following problem:

GIVEN a set of uncertain multiplicity counter (inter-arrival time) measurements, T_M; FIND the "best" estimate $\hat{\Theta}$ of the unknown fission source parameters, Θ.

From a statistical perspective, we would like to estimate the posterior distribution of source parameters Θ given the entire inter-arrival data set T_M or $\Pr[\Theta|T_M]$. Applying Bayes' theorem we have that

$$\Pr[\Theta|T_M] = \frac{\Pr[T_M|\Theta]\times\Pr[\Theta]}{\Pr[T_M]} \tag{23}$$

Due to the sequential nature of our problem, that is, the neutron multiplicity counter measures each neutron arrival time individually—neutron-by-neutron; we require a *sequential version*.

A. Sequential Bayesian Processor

It can be shown [32] that a sequential Bayesian solution can be developed for the posterior. Starting with the first term of Eq. 23 and applying Bayes' rule we have

$$\Pr[T_M|\Theta] = \Pr[\tau_M, T_{M-1}|\Theta] = \Pr[\tau_M|T_{M-1},\Theta]\times\Pr[T_{M-1}|\Theta] \tag{24}$$

and for the denominator term we have

$$\Pr[T_M] = \Pr[\tau_M, T_{M-1}] = \Pr[\tau_M|T_{M-1}]\times\Pr[T_{M-1}] \tag{25}$$

Substituting Eqs. 24 and 25 into Eq. 23 and grouping terms, we obtain

$$\Pr[\Theta|T_M] = \underbrace{\left(\frac{\Pr[\tau_M|T_{M-1},\Theta]}{\Pr[\tau_M|T_{M-1}]}\right)}_{W(\tau_m)}\times\underbrace{\left(\frac{\Pr[T_{M-1}|\Theta]\times\Pr[\Theta]}{\Pr[T_{M-1}]}\right)}_{\Pr[\Theta|T_{M-1}]} \tag{26}$$

or

$$\Pr[\Theta|T_M] = W(\tau_m)\times\Pr\left[\Theta|T_{M-1}\right] \tag{27}$$

which is the sequential form of the *posterior distribution*. If we further assume that the inter-arrivals are Markovian with the current arrival depending only on the previous, that is, $(\tau_m, T_{M-1}) \longrightarrow (\tau_m, \tau_{m-1})$, then we have the desired expression for sequentially propagating the posterior as

$$\Pr[\Theta|\tau_m] = W(\tau_m)\times\Pr\left[\Theta|\tau_{m-1}\right] \tag{28}$$

where

$$W(\tau_m) = \frac{\Pr\left[\tau_m|\tau_{m-1},\Theta\right]}{\Pr\left[\tau_m|\tau_{m-1}\right]} \tag{29}$$

Here we assumed that the parameter vector is a random constant Θ with no associated dynamics to construct a sequential Bayesian processor. However, in the real-world case, it is clear that when measuring neutron inter-arrivals from an unknown source, then there can easily be variations or uncertainties associated with each parameter. Perhaps a more reasonable model for these parametric variations is the *random-walk/constant* introduced in the previous section [32]. That is, we know in continuous-time that the walk is

given by $\frac{d}{dt}\Theta(\tau) = w_\Theta(\tau)$ and by taking first differences to approximate the derivative we can obtain a sampled-data representation [32] as

$$\Theta(\tau_m) = \Theta(\tau_{m-1}) + \triangle\tau_m \; w_\Theta(\tau_{m-1}) \tag{30}$$

where $\triangle\tau_m := \tau_m - \tau_{m-1}$ for the parametric uncertainty included in $\Theta(\tau_m) \sim \mathcal{N}(m_\Theta(\tau_m), R_{\Theta,\Theta}(\tau_m) + R_{w_\theta w_\theta}(\tau_m))$, since $w_\Theta \sim \mathcal{N}(0, R_{w_\theta w_\theta}(\tau_m))$. The variations of each Θ-parameter can be controlled by its initial guess $\Theta(\tau_0)$ and variance, $R_{\Theta,\Theta}(\tau_m) + R_{w_\theta w_\theta}(\tau_m)$.

Physically we have that $R_{w_\theta w_\theta} = \text{diag}[\sigma^2_{MM} \quad \sigma^2_{k_{\text{eff}}k_{\text{eff}}} \quad \sigma^2_{\epsilon\epsilon} \quad \sigma^2_{\lambda^{-1}\lambda^{-1}}]$ and the subsequent search can include a bounded uniform variate for each initial value, $\Theta_0 = \mathcal{U}[a, b]$ enabling a more pragmatic approach to modeling the parameter set with their individual accompanying uncertainties. The statistics of the *random walk/constant* model are given by the sequential *Gauss-Markov structure*

$$\begin{aligned} \Theta(\tau_m) &= \Theta(\tau_{m-1}) + \triangle\tau_m \; w_\Theta(\tau_{m-1}) && \text{[STATE]} \\ m_\Theta(\tau_m) &= m_\Theta(\tau_{m-1}) && \text{[MEAN]} \\ R_{\Theta\Theta}(\tau_m) &= R_{\Theta\Theta}(\tau_{m-1}) + R_{w_\theta w_\theta}(\tau_{m-1}) && \text{[VARIANCE]} \end{aligned} \tag{31}$$

With this model in mind, we re-derive the sequential Bayesian processor as before starting with the *batch* approach. We would like to estimate $\Theta(\tau_m)$ with the complete parameter set defined by $\Theta_M := \{\Theta(\tau_0), \Theta(\tau_1) \cdots, \Theta(\tau_M)\}$. The batch posterior is given by Bayes' theorem as before

$$\Pr[\Theta_M|T_M] = \frac{\Pr[T_M|\Theta_M] \times \Pr[\Theta_M]}{\Pr[T_M]} \tag{32}$$

The first term can be decomposed by applying Bayes' rule as

$$\begin{aligned} \Pr[T_M|\Theta_M] &= \Pr[\tau_m, T_{M-1}|\Theta(\tau_m), \Theta_{M-1}] \\ &= \Pr[\tau_m|T_{M-1}, \Theta(\tau_m), \Theta_{M-1}] \\ &\times \Pr[T_{M-1}|\Theta(\tau_m), \Theta_{M-1}] \end{aligned} \tag{33}$$

The inter-arrival time τ_m is independent Θ_{M-1} and T_{M-1}, so that the first term in Eq. 33 becomes $\Pr[\tau_m|\Theta(\tau_m)]$, while the second term simplifies to $\Pr[\tau_{M-1}|\Theta_{M-1}]$, since the parameter vector $\Theta(\tau_m)$ is assumed independent of the past measurement data. Therefore, we have

$$\Pr[T_M|\Theta_M] = \Pr[\tau_m|\Theta(\tau_m)] \times \Pr[T_{M-1}|\Theta_{M-1}] \tag{34}$$

The second term of Eq. 32 can be decomposed similarly as

$$\Pr[\Theta_M] = \Pr[\Theta(\tau_m), \Theta_{M-1}] = \Pr[\Theta(\tau_m)|\Theta_{M-1}] \times \Pr[\Theta_{M-1}] \tag{35}$$

and the decomposition of $\Pr[T_M]$ in the denominator is given in Eq. 24 above.

Substituting these relations (Eqs. 24, 34, 35) into Eq. 32, and assuming a Markovian process as before, we obtain

$$\begin{aligned} \Pr[\Theta_M|T_M] &= \Big(\Pr[\tau_m|\Theta(\tau_m)] \times \Pr[T_{M-1}|\Theta_{M-1}]\Big) \\ &\times \Big(\Pr[\Theta(\tau_m)|\Theta(\tau_{m-1})] \times \Pr[\Theta_{M-1}]\Big) \\ &/ \quad \Pr[\tau_m|\tau_{m-1}] \times \Pr[T_{M-1}] \end{aligned} \tag{36}$$

Now grouping the terms the desired posterior distribution becomes

$$\begin{aligned} \Pr[\Theta_M|T_M] &= \Big(\frac{\Pr[\tau_m|\Theta(\tau_m)] \times \Pr[\Theta(\tau_m)|\Theta(\tau_{m-1})]}{\Pr[\tau_m|\tau_{m-1}]}\Big) \\ &\times \Big(\frac{\Pr[T_{M-1}|\Theta_{M-1}] \times \Pr[\Theta_{M-1}]}{\Pr[T_{M-1}]}\Big) \end{aligned} \tag{37}$$

or simply (replacing $\tau_m \longrightarrow T_M$)

$$\Pr[\Theta_m|T_m] = W(\tau_m) \times \Pr[\Theta_{m-1}|T_{m-1}] \tag{38}$$

where

$$W(\tau_m) := \frac{\Pr\left[\tau_m|\Theta(\tau_m)\right] \times \Pr[\Theta(\tau_m)|\Theta(\tau_{m-1})]}{\Pr\left[\tau_m|\tau_{m-1}\right]} \tag{39}$$

which is the sequential Bayesian solution for the dynamic parametric model (random walk/constant). A particle filter is an implementation of this recursion [32], [33].

B. Particle Filter for Fission Processes

Particle filtering is a technique that evolves from the "importance sampling" approach to statistical sampling of data. The key idea is to select particles (or parameters in our problem) from the regions of highest probabilities or equivalently regions of *highest importance*. Once the resulting importance weight is determined, the desired posterior distribution is approximated by a non-parametric probability mass function (*PMF*) as

$$\hat{\Pr}[\Theta_M|T_M] = \sum_i \mathcal{W}_i(\tau_m) \; \delta\Big(\Theta(\tau_m) - \Theta_i(\tau_m)\Big) \tag{40}$$

where $\mathcal{W}_i$ is the *normalized* weighting function given as the ratio of the posterior at inter-arrival time τ_m and the designed importance distribution q as

$$\begin{aligned} W_i(\tau_m) &:= \frac{\Pr[\Theta(\tau_m)|T_m]}{q[\Theta(\tau_m)|T_m]} = \frac{\Pr[\tau_m|\Theta(\tau_m)]}{q[\tau_m|\Theta(\tau_m)]} \\ &\times \frac{\Pr[\Theta(\tau_m)]}{q[\Theta(\tau_m)|\Theta_{m-1}, \tau_m]} \end{aligned} \tag{41}$$

The normalized weight is simply

$$\mathcal{W}_i(\tau_m) := \frac{W_i(\tau_m)}{\sum_i W_i(\tau_m)} \tag{42}$$

The "bootstrap" processor is the most popular technique [32]. Here the proposal is selected as the *transition* prior and the weighting function becomes simply the likelihood

$$W(\tau_m) = W(\tau_{m-1}) \times \Pr[\tau_m|\Theta(\tau_m)] \tag{43}$$

With these relations in mind, the sequential Bayesian processor can be developed for our problem. We start with the basic bootstrap technique to estimate the unknown source parameters that will eventually become part of the log-likelihood decision function. Initially, we assume the prior distributions are uniformly distributed with bounds selected over some pragmatic intervals $\mathcal{U}(a, b)$. The dynamic parameter updates are given by the random-walk/constant model of Eq. 30 driven by zero-mean, Gaussian noise with covariance $R_{w_\Theta w_\Theta}$ with initial mean (constant) $\Theta(\tau_0)$ and corresponding parametric covariance $R_{\Theta\Theta}(\tau_m)$. The likelihood distribution embeds the "fission physics" of Eq. 3.

The measurement prediction model is based on the current parameter estimates and its analytic time-interval likelihood distribution. As a new neutron inter-arrival τ_m is available at the output of the multiplicity counter, the predicted parameter estimate $\hat{\Theta}(\tau_m|\tau_{m-1})$ is made and provided as input to the physics likelihood probability distribution $\mathcal{L}_\Theta(\hat{\Theta}(\tau_m)) := \Pr(\tau|\Theta)$ for the prediction.

After the predicted inter-arrival $\hat{\tau}_m$ is available, the update and resampling steps are performed and the next measurement is awaited. The bootstrap algorithm performs the following steps shown in Table 1.

Table 1. Bootstrap SIR PF for Fission Processing

PRIOR

$$\Theta_i(\tau_0) \sim \Pr(\Theta(\tau_0)) \rightarrow \mathcal{U}(\theta_a, \theta_b);\ W_i(\tau_0) = \frac{1}{N_p}$$

$i = 1, \cdots, N_p$ [sample]

PREDICTION

$\Theta_i(\tau_m) = \Theta_i(\tau_{m-1}) + w_i(\tau_{m-1}); w_i \sim \mathcal{N}(0, R_{w_i w_i})$ [transition]

$\hat{\tau}_m = \max_{\tau \in T} \mathcal{L}[\tau_m]$ [Measurement Pred.]

Weight (Likelihood) Update:

$W_i(\tau_m) = \mathcal{L}\left[\tau_m|\Theta_i(\tau_m)\right]$ [weight/likelihood]

Weight normalization:

$$\mathcal{W}_i(\tau_m) = \frac{W_i(\tau_m)}{\sum_{i=1}^{N_p} W_i(\tau_m)}$$

RESAMPLING DECISION

$$\hat{N}_{eff} = \frac{1}{\sum_{i=1}^{N_p} W_i^2(\tau_m)} \quad \text{[effective samples]}$$

$$\hat{N}_{eff} = \begin{cases} \text{Resample} & \leq N_{thresh} \\ \text{Accept} & > N_{thresh} \end{cases}$$

RESAMPLING

$$\hat{\Theta}_i(\tau_m) \Rightarrow \Theta_i(\tau_m)$$

POSTERIOR DISTRIBUTION

$$\hat{\Pr}(\Theta(\tau_m)|T_m) \approx \sum_{i=1}^{N_p} \mathcal{W}_i(\tau_m)\delta(\Theta(\tau_m) - \hat{\Theta}_i(\tau_m))$$

V. SNM DETECTION AND ESTIMATION: SYNTHESIZED DATA

In this section we investigate the performance of the sequential Bayesian detector/estimator on simulated data (see Sec. II for details). We assume a set of parameters to statistically synthesize a set of arrivals used to calculated the inter-arrivals $\{\tau_m\};\ m = 0, 1, \cdots, M$. We start with the results from the sequential Bayesian detection algorithm that incorporates the Bayesian parameter estimator as part of its inherent structure as illustrated in Fig. 1. Note that the unknown source parameter vector to be estimated has the following physical parameters (from the inter-arrival *PDF*): $\Theta := [m_{\rm S}\ k_{\rm eff}\ \epsilon\ \lambda^{-1}]^T$— the mass of the source, effective multiplication ($k_{\rm eff}$), detector efficiency and diffusion time, respectively.

The bootstrap *PF* was applied to a synthesized neutron arrival sequence shown in Fig. 1. We note from the figure that the inter-arrival data is processed by the sequential Bayesian estimator to provide predicted estimates of the source parameters, $\hat{\Theta}(\tau_m|\tau_{m-1})$. These estimates are then input to the physics-based likelihood to predict the corresponding *PDF* which is a part of the log-likelihood decision function. These predicted parameters are also provided directly as individual inputs to the log-likelihood decision function. Once the decision function is calculated at τ_m, it is compared to the thresholds to "decide" whether or not a threat is present. If so, the alarm is initiated. If not, another measurement is processed on arrival (take more data).

There exists a variety of metrics that can be applied to evaluate detection performance ranging from confusion matrices to sophisticated statistical hypothesis tests [39], but perhaps the most basic and most robust method is the calculation of the receiver operating characteristic curve. The *ROC* curve is simply a graph of detection (P_{DET}) versus false alarm (P_{FA}) probabilities parameterized by threshold, $\mathcal{T}$ with *perfect* performance occurring when $P_{DET} = 1$ and $P_{FA} = 0$. The *ROC* curve, provides all of the fundamental information from which most other metrics are derived. Thus, there are many individual metrics that can be extracted directly from a *ROC* curve including sensitivity, specificity, cost/benefit analysis along with a set of specific features like area under-the-curve (*AUC*) and minimum probability of error (MinE) [39].

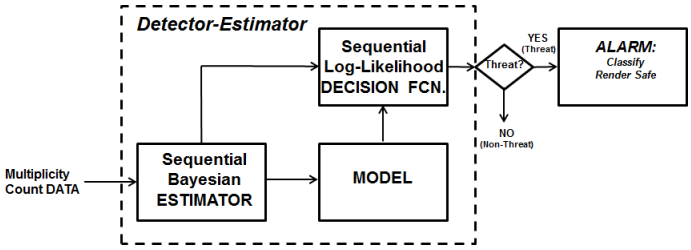

Fig. 1. Fission detection/estimation processing: Count data, fission parameter estimation, sequential detection and alarm.

As mentioned in the Sec. III, it is necessary to calculate a *ROC* curve to select an operating point (detection and false-alarm probabilities) to calculate the sequential thresholds. In order to generate the *ROC*, we synthesize an ensemble of 30-members each consisting of 100-arrivals selected directly from a Monte Carlo simulation data set using the following source (uranium) parameters: $m_{\mathrm{S}} = 25$ kg, $k_{\mathrm{eff}} = 0.9$; $\epsilon = 0.03$; $\lambda^{-1} = 0.01$. We chose to use a signal-to-noise ratio (*SNR*) of 6.9 dB defined by the 10 log-ratio of the signal energy to noise energy (variance). The local *ROC* for each member realization was estimated and then the *average ROC* was used for the calculation. Performance metrics such as *AUC* are also calculated to assess detection performance (*AUC*=0.95). For "perfect" performance, the detection probability is unity and false-alarm probability zero corresponding to an *AUC* of unity. The optimum operating point (P_{FA}, P_{DET}) is calculated by minimizing the Bayes' risk (see [39] for details) yielding a detection probability of 92% for a false alarm probability of 0.1% at this *SNR*. Substituting these values into the threshold calculation of Eq. 8 and taking the natural logarithm give the thresholds $(\ln \tau_0, \ln \tau_1)$ as $(-2.48, 7.23)$. A typical realization of sequential Bayesian detection results for a member of the ensemble is shown in Fig. 2 showing the decision function exceeding the upper threshold thereby indicating a threat and subsequent alarm. The *ROC* curve enables us to evaluate the performance of the sequential detection algorithm at various *SNR*s. This performance is what we would expect to achieve at the selected thresholds. Test data is used to verify these predictions. So we see that the detection performance is quite reasonable at these *SNR*s indicating a feasible Bayesian detector design.

Only 100 inter-arrivals were investigated to observe the feasibility of this approach. We observe the uncertainty of the inter-arrivals caused by the randomness of the fission process. Using the predictive sampling approach described in the previous section, we compare the predicted inter-arrivals to the measured along with their difference (residual) as illustrated in Fig. 3. Here we see that the prediction is reasonable. Next the physical parameters were estimated with the results shown in Fig. 4. Again the parameter estimates are quite reasonable for this realization at 6.9 dB. For parameter estimation performance metrics, we use the average *RMS*-error (absolute/relative errors) of each estimator to give: mass (*RMS*-error = $0.03/0.28\%$ kg); effective multiplication (*RMS*-error = $0.0025/0.12\%$); efficiency (*RMS*-error = $0.00006/0.20\%$); and inverse diffusion time (λ) (*RMS*-error = $0.00004/0.4\%$ $\mu\mathrm{s}^{-1}$).

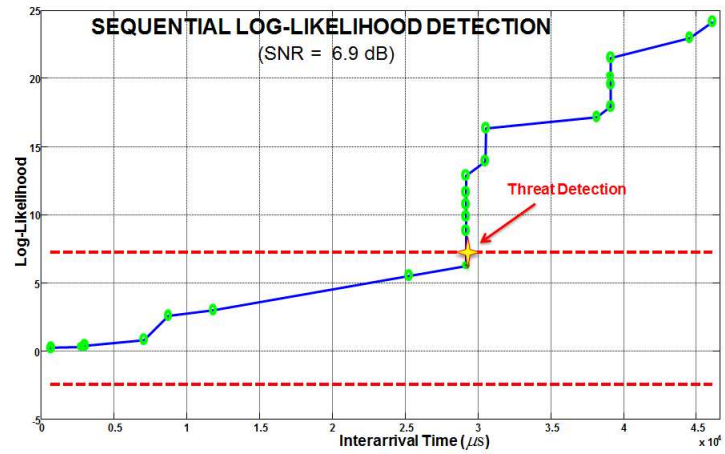

Fig. 2. Sequential Bayesian (log-likelihood) threat detection for (P_{FA}, P_{DET})=(0.0007, 0.916) and thresholds at $(\ln \tau_0, \ln \tau_1)$=$(-2.48, 7.23)$.

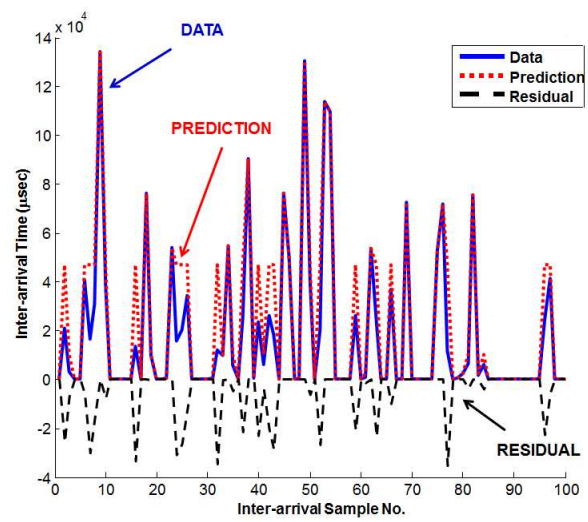

Fig. 3. Raw (solid line) and predicted multiplicity counter inter-arrivals (dotted line) using prediction sampling scheme along with the resulting residuals (dashed line) (*SNR*=6.9dB).

Note that the true parameter value is shown as the line (dashed) and both the maximum a-posterior (*MAP*) and conditional mean (*CM*) estimates are shown (arrows) on the plots. They appear to track the physical parameters quite well (small *RMS* errors) for this realization.

A pragmatic statistical test (in general) for the bootstrap processor is that the residual sequence should be zero-mean and uncorrelated. In statistical signal processing, a standard statistical test to determine whether or not a processor is operating properly is to test whether the residual sequence (difference between the measurement and the processor "predicted" measurement) has a small error variance. Specifi-

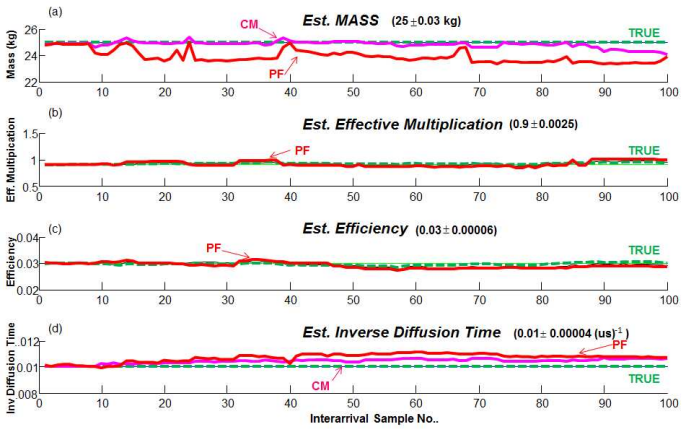

Fig. 4. Sequential Bayesian parameter estimates (arrows) and absolute *RMS* error for a *SNR*=6.9dB: (a) Mass (25 ± 0.03 kg). (b) Effective multiplication ($k_{\text{eff}} = 0.90 \pm 0.0025$). (c) Efficiency ($0.03 \pm 0.00006$). (d) Inverse diffusion time ($\lambda = 0.01 \pm 0.00004\ \mu s^{-1}$).

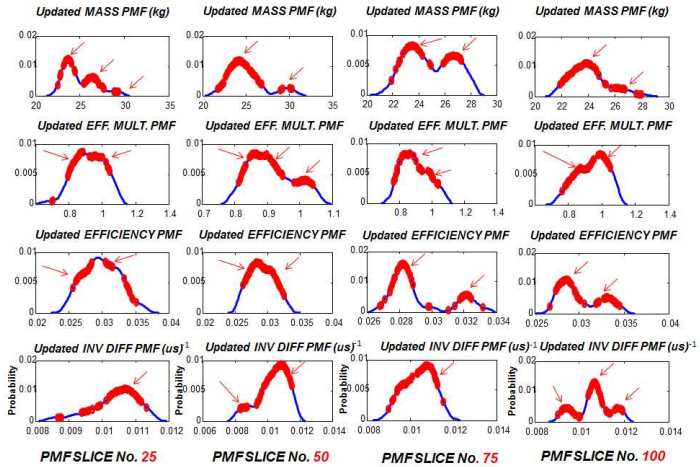

Fig. 5. Posterior *PMF* of various slices ($25, 50, 75, 100$ interarrivals) illustrating the multimodal (arrows) nature of the distributions as well as the coalescing of particles (circles) in highest probability regions.

cally for sequential algorithms, the one-step prediction error (or residual) has the property that the prediction error or equivalently the residual sequence should be zero-mean and uncorrelated (or white). Pragmatically, residuals that satisfy these properties ensure that the embedded model-based processor has "removed" all correlation (of the model) from the data. The remaining measurement (residual) is uncorrelated or equivalently provides no further information about the signal (or model). These properties ensure two conditions: (1) the underlying model is correct; and (2) no further signal information remains. In fact, it can be shown that these conditions are a result of the orthogonality condition of the data and its prediction [37]. Indeed this is the case, since the zero-mean test shows the residual mean to be smaller than the calculated bound (< 0.245) and uncorrelated with only 0.0% of the correlation samples lying outside of the boundaries (5% or more are considered correlated). The weighted sum-squared residual (*WSSR*) is a chi-squared statistic that tests the hypothesis of whether or not the residual sequence is statistically uncorrelated or not. It establishes a threshold and requires the underlying decision function to be below for whiteness (not correlated). More of the *WSSR* details are available in [32], [37] along with the corresponding references. The *WSSR* statistic confirms the whiteness with its decision function lying below the threshold of 51.4. Therefore, from a local perspective, the processor appears to be performing well. However, in order to confirm this statement, we must perform a sequence of at least 100 realizations and calculate ensemble statistics (not shown). Finally we show a set of for snapshots (or slices) at various inter-arrival samples $(25, 50, 75, 100)$ throughout the simulation. Note that each is a slice of the $3D$ *PMF*s shown for all of the physical parameters. Also note how the estimated particles coalesce (as expected) about the highest probability regions which are annotated by the arrows in Fig. 5.

VI. SUMMARY

We have developed a sequential Bayesian approach to the fission detection problem based on a theoretical likelihood *PDF* capturing the underlying physics of the fission process [20],[21]. We developed the sequential detection processor based on a joint probability distribution of inter-arrivals and unknown source parameters using a combined random walk and random constant model of the parameter uncertainties [32]. This model was embedded along with the physics-based likelihood into both sequential detection and parameter estimation processors. The processor performed reasonably on synthesized Monte Carlo data resulting in a successful threat (*SNM*) detection achieving a detection/false alarm probability of $P_{DET} = 91.7\%$ at a $P_{FA} = 0.07\%$ in comparison to a similar theoretical approach applied to γ-ray detection of $P_{DET} = 95\%$ at a $P_{FA} = 2\%$ [7],[9]. The physics parameter estimates were also quite reasonable with relative *RMS* errors less than 0.5% indicating a feasible solution to the fission detection problem. Performance metrics for both detection and parameter estimation processors were developed and assessed implying that a combination of both sequential Bayesian γ-ray [7],[9] and neutron detection schemes have the potential of providing both a timely and accurate threat detection capability.

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