LAWRENCE
LIVERMORE
NATIONAL
LABORATORY

LLNL-CONF-667596

WIP: Probabilistic Modeling of GPS-based Capabilities

V. Standley, E. Boucheron, R. Kirkwood, B. Norman

February 23, 2015

SpringSim 2015
Alexandria, VA, United States
April 12, 2015 through April 15, 2015

Disclaimer

This document was prepared as an account of work sponsored by an agency of the United States government. Neither the United States government nor Lawrence Livermore National Security, LLC, nor any of their employees makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States government or Lawrence Livermore National Security, LLC. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States government or Lawrence Livermore National Security, LLC, and shall not be used for advertising or product endorsement purposes.

LLNL-CONF-667596

WIP: Probabilistic Modeling of GPS-based Capabilities

Vaughn H. Standley
National Nuclear Security Administration[1]

Edward A. Boucheron
Sandia National Laboratories[2]

Robert K. Kirkwood
Lawrence Livermore National Laboratory[3]

Benjamin E. Norman
Los Alamos National Laboratory[4]

ABSTRACT

In terms of the number of satellites in view from a ground level device or event, the global coverage provided by the GPS constellation is demonstrated to be well approximated by assuming that the satellites are randomly distributed around the earth. Analysis using GPS almanac data indicates that this is a good approximation to within about 10% accuracy. An analytic model relying on this approximation enables parametric estimation of cost, performance, and risk of GPS-based capabilities and is a means to help validate more sophisticated performance models where the average statistics representing global coverage is the dominant factor, a work in progress (WIP).

INTRODUCTION

For more than 50 years, the National Nuclear Security Administration and its predecessors have sponsored work at the National Nuclear Laboratories to develop and fabricate instruments to detect nuclear detonations from space. Synergies with space-based navigation led to these instruments being hosted on GPS satellites beginning in the 1980's, since both navigation and detonation detection missions require global coverage and the use of clocks to determine location. While coverage provided by these systems is normally assessed using sophisticated orbital ephemeris-based models[1] (e.g. Satellite Tool Kit, Satellite Orbit Analysis Program), it is easy to provide reasonable estimates with a simple analytic model. Estimates from this method can be used to help inform acquisition decisions and as a validation check of performance estimates provided by more complicated models. The method shown here calculates the probability that a specific number of GPS-based instruments will receive signals from a nuclear detonation or, conversely, that radio frequency receivers on the ground will receive signals from a specific number of GPS satellites. This method, not previously used, is based on modeling satellites' sensing behavior with a single parameter – the instrument Look-Angle which can be used to generate a highly aggregated system representation.

© 1994, 1995, 1998, 2002, 2009, 2011, 2013 by ACM, Inc. Permission to copy and distribute this document is hereby granted provided that this notice is retained on all copies, that copies are not altered, and that ACM is credited when the material is used to form other copyright policies.

[1] 1000 Independence Ave. SW., Washington DC, 20585, vaughn.standley@nnsa.doe.gov

[2] P.O. Box 5800, Albuquerque, NM 87185, edward.a.boucheron@aero.org

[3] P.O. Box 808, Livermore, CA 94551, kirkwood1@llnl.gov

[4] P.O. Box 1663, Los Alamos, NM 87545, bnorman@lanl.gov

MAXIMUM LOOK ANGLE (MLA):

The sensitivity of satellite-based instruments to events on the surface of the earth can be represented very simply in terms of a maximum "Look-Angle" that does not depend on details of local terrain and represents a spherically symmetric earth and atmosphere. Maximum Look-Angle (MLA) is the greatest angle off zenith to where satellite based instruments are still sufficiently sensitive to detect a surface event of specified characteristic signals, such as a nuclear explosion. Look-Angles are used because they take into account the attenuation of propagating signals through the atmosphere. This attenuation is least when observing directly down to the earth, and greater when looking obliquely toward the earth's horizon. Figure 1 illustrates the relationship between the event, observing instrument, earth, and subtended angles. Maximum Look Angles are computed by the sensor developers based on analysis and testing. For example, the implied Maximum Look Angle associated with a hand-held GPS receiver is almost 90 degrees, meaning that an earth based observer can receive signals from all satellites above the earth horizon.

For purposes of a simple mathematically abstracted representation, the relationship between Maximum Look Angle and the number of average satellites in-view is treated as having a strictly geometric relationship, calculated by first determining the fraction of the earth over which any one instrument can detect an event or signals from the satellite can be received. The fraction of the earth area viewed by a satellite deployed instrument, based on the surface area of a unit dome is:

$$\textit{Fraction of Earth Area in View} = \frac{(1-\cos\beta)}{2}$$

The Law of Sines is used to derive β, which is given by:

$$\beta = cos^{-1}\left(\frac{r_e}{r_s} cos\,\theta\right) - \theta$$

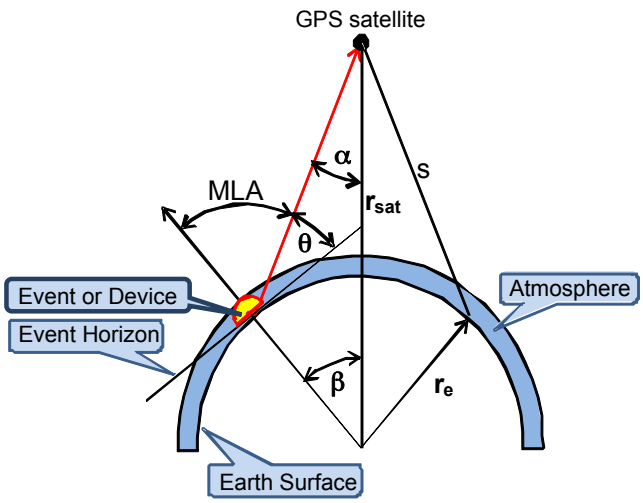


Figure 1. Relationship between Maximum Look Angle (MLA), Elevation Angle (θ), Satellite Cone Angle (α), and Earth Central Angle (β).

ESTIMATING PROBALISTIC COVERAGE USING MLA:

For GPS-based capabilities, the ratio of earth and satellite orbital radii, $r_{earth}/r_{satellite}$ is equal to 6378km/(20200km + 6378km) = 0.24. For N total satellites in a constellation, each with a detonation detection instrument or navigation transmitter onboard, assumed to be randomly and uniformly distributed around the sphere of the earth, the average number of satellites in view can be derived as the fraction of the earth area covered by each sensor, multiplied by N:

$$Average\ Satellites\ in\ View = N\frac{(1-\cos\beta)}{2}$$

After inserting β into the above equation and replacing θ with 90-MLA, one obtains a direct relationship between average number of satellites in view over the whole earth and maximum look-angle:

$$Equation\ 1$$

$$Average\ Satellites\ in\ View =$$

$$\frac{N}{2}\left(1 - sin\left(cos^{-1}\left(\frac{r_e}{r_s} sin(MLA)\right) + (MLA)\right)\right)$$

Our analysis uses the result of Equation 1 rounded to the nearest integer as an estimate of the actual number of satellites in view so that the rest of the calculations can be done using a hypergeometric distribution. We will subsequently show by comparisons with more precise calculations (discussed below) that this assumption is reasonably accurate for the parameters of the GPS constellation. A simple random model can be based solely on mixing the average number of satellites in-view from Equation 1 with those not in view. Confining ourselves first to the half-earth case, we use the hypergeometric distribution with population size N = 12, sample size n = 12, the number of successes in the sample = k, and number of successes in the population = K = Equation 1. So for a specific example where MLA = 90 degrees, K = 9. The mean of the hypergeometric distribution is n*K/N, which is for this case 12*9/12 = 9. While not particularly useful, the half-earth case is instructive because it shows that the hypergeometric distribution returns a probability of one for k = K and zero otherwise. That is, the number of satellites in view as calculated by Equation 1 is always true. A whole-earth, 24-satellite GPS constellation can be thought of as twice a half-earth constellation with N = 2*12 = 24 and number of successes twice the number for the half-earth case. However, we keep the sample size n equal to 12 and thus the hypergeometric will return results other than simply zero or one. We use twelve because it corresponds to the maximum number of satellites that can ever be in view. Conversely, it can be thought of as how many GPS-based detonation detection instruments are in view of an event on the ground if the instruments had the same look-angle for the event.

Consider the example of a GPS handheld receiver that has the equivalent of a satellite sensor maximum look angle of 85 degrees. The integer number of satellites in view for half-earth according to Equation 1 is equal to 8. For the whole earth, there is an average of 2*8=16 in view. For reasons explained, we keep the sample size n equal to 12. The probability that exactly eight satellites will be in view using integer values derived from Equation 1 is:

$$P(X=8) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} = \frac{\binom{16}{8}\binom{24-16}{12-8}}{\binom{24}{12}} = \frac{12870 * 70}{2704156} = 0.333$$

Where N=24, K=16, n=12, k=8 and $\binom{a}{b}$ is the binomial coefficient for integers a and b. The result above makes sense because the ratio of the average satellites in view to the total number of satellites, 8/24, is numerically the same. Cumulative probability is more useful because, for example, handheld receivers need information from at least four satellites to compute a three-dimensional coordinate. The cumulative probability associated with a minimum number of satellites in view is given by

$$P(X \geq k) = 1 - \sum_{0}^{k-1} \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Using this model, the probability that at least four satellites are in view of a GPS hand-held receiver for the example above is:

$$P(X \geq 4) = 1 - \left(\frac{\binom{16}{0}\binom{24-16}{12-0}}{\binom{24}{12}} + \frac{\binom{16}{1}\binom{24-16}{12-1}}{\binom{24}{12}} + \frac{\binom{16}{2}\binom{24-16}{12-2}}{\binom{24}{12}} + \frac{\binom{16}{3}\binom{24-16}{12-3}}{\binom{24}{12}} \right)$$

$$= 1 - (0 + 0 + 0 + 0) = 1$$

This result illustrates a limitation of using a discrete random model. Under some circumstances the results will either be exactly one or zero, which might leave some doubt about how different the true value will be. However, this concern can easily be addressed by calculating the result for slightly different coverage circumstances. For example, the probability that at least five satellites are in view is given by:

$$P(X \geq 5) = 1 - \left(0 + 0 + 0 + 0 + \frac{1820 * 1}{2704156}\right) = 0.999$$

That is, there is a 99.9% chance of finding one more than the minimum number of GPS satellites needed to compute a location from anywhere on the earth. Thus, the true value for at least four satellites in view is somewhere between 0.999 and 1, which should be satisfactory knowledge under most circumstances.

CONFIRMING ESTIMATES USING ALMANAC DATA:

A method of modeling the number of GPS satellites with sensors in view of an event was also performed by a Monte Carlo sampling of GPS orbital almanac data. This method directly calculates the average number of satellites in view and was used to independently validate the simple hypergeometric analytical model developed. To this end, a dynamic model of the GPS constellation orbits has been represented by randomly sampling an almanac of satellite elevations over time using ephemeris data for a given GPS constellation. GPS satellites exhibit an orbital period of one-half a sidereal day, i.e., 11 hours and 58 minutes. The orbits are arranged so that at least six satellites are always within line of sight from almost everywhere on Earth's surface. Orbiting at an altitude of approximately 20,200 km; orbital radius of approximately 26,600 km, each SV makes two complete orbits each sidereal day, repeating the same ground track each day. Trimble Planning software, version 2.90, was used to generate lookup tables for the elevation angles of satellites in a constellation of 24 GPS satellites every minute over a 72 hour period for a set of assigned "ground station" latitudes. Since the GPS satellite ground tracks repeat each day, use of a three day lookup table as a repeating pattern over time is deemed a reasonable approximation of true constellation behavior. Further, it is assumed that the southern hemisphere observations of satellite elevations will be a mirror image of those in the northern hemisphere. The latitudes chosen were 0.0, 7.5, 22.5, 30, 40, 60, 70 and 90 degrees, all at 77 degrees longitude. Associated with each value of latitude is a section of the hemisphere of the earth surface, each of which covers a fraction of the earth's surface area respectively as shown in Table 1.

Table 1: Ground station latitudes and range of latitudes assumed to be represented along with represented fraction of the Earth's surface and cumulative fraction from Equator to North Pole

Ground Station Latitude (degrees)	Latitude Range Represented (degrees)	Fraction of Earth Surface Represented	Cumulative Fraction Represented from 0 to 90 degrees Latitude
90	80-90	0.02	1.00
70	65-80	0.08	0.98
60	50-65	0.14	0.90
40	36.5-50	0.17	0.76
30	27.75-36.5	0.13	0.59
22.5	15-27.75	0.21	0.46
7.5	3.75-15	0.19	0.25
0.0 (Equator)	0-3.37	0.06	0.06

An almanac of satellite elevations, output at one minute intervals over a 72 hour period, as seen from each of these latitudes is then input as a separate 'sheet' in one Excel spreadsheet book. A random number generator is used to select one of the 4321 time slots that an event is assigned to take place. The table entries contain all the visible satellite elevations above the horizon at that location. It is then a simple matter to subtract these elevations from 90 degrees to convert to equivalent satellite Look Angle represented at that location. Next, the spreadsheet randomly assigns a satellite number, 1 to 24 to the table entries. The converted Look Angles associated with satellites are then compared to the assumed sensor Maximum Look Angle to determine whether the satellite sensor "sees the event" or not. Of course, the converse is represented as well – whether a ground based sensor successfully receives a signal from the satellite. If the event is seen or the signal is received on the ground, the detection probability, here assumed to be one, is then multiplied by the input satellite/sensor reliability, say 0.98. Thus, each satellite deemed to have seen an event or had its timing signal successfully received has an individual detection probability of 0.98. Next, the cumulative probabilities are calculated for the cases of one, two and so on, up to 14 satellites detecting an event. The cumulative probability is defined as

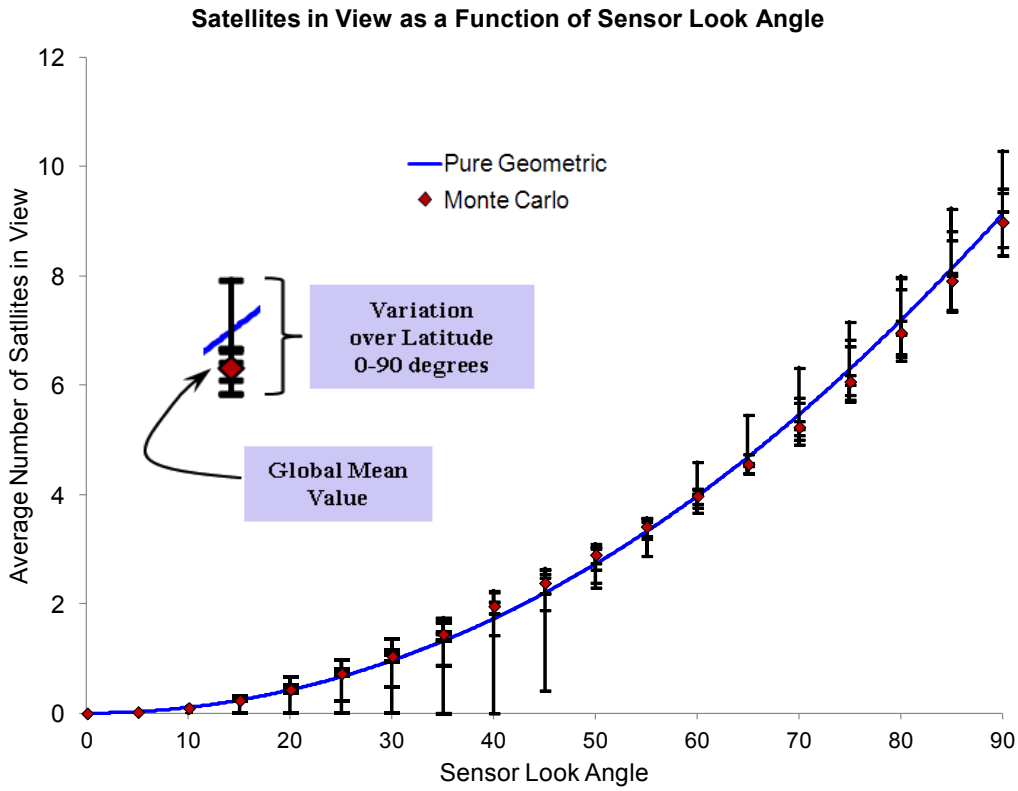


Figure 2. Satellites in-view vs. Maximum Look Angle for a 24-Satellite GPS Constellation

$$\textit{Cumulative Probability of Detection} = Pd_n + Pd_{n+1} - Pd_n * Pd_{n+1}$$

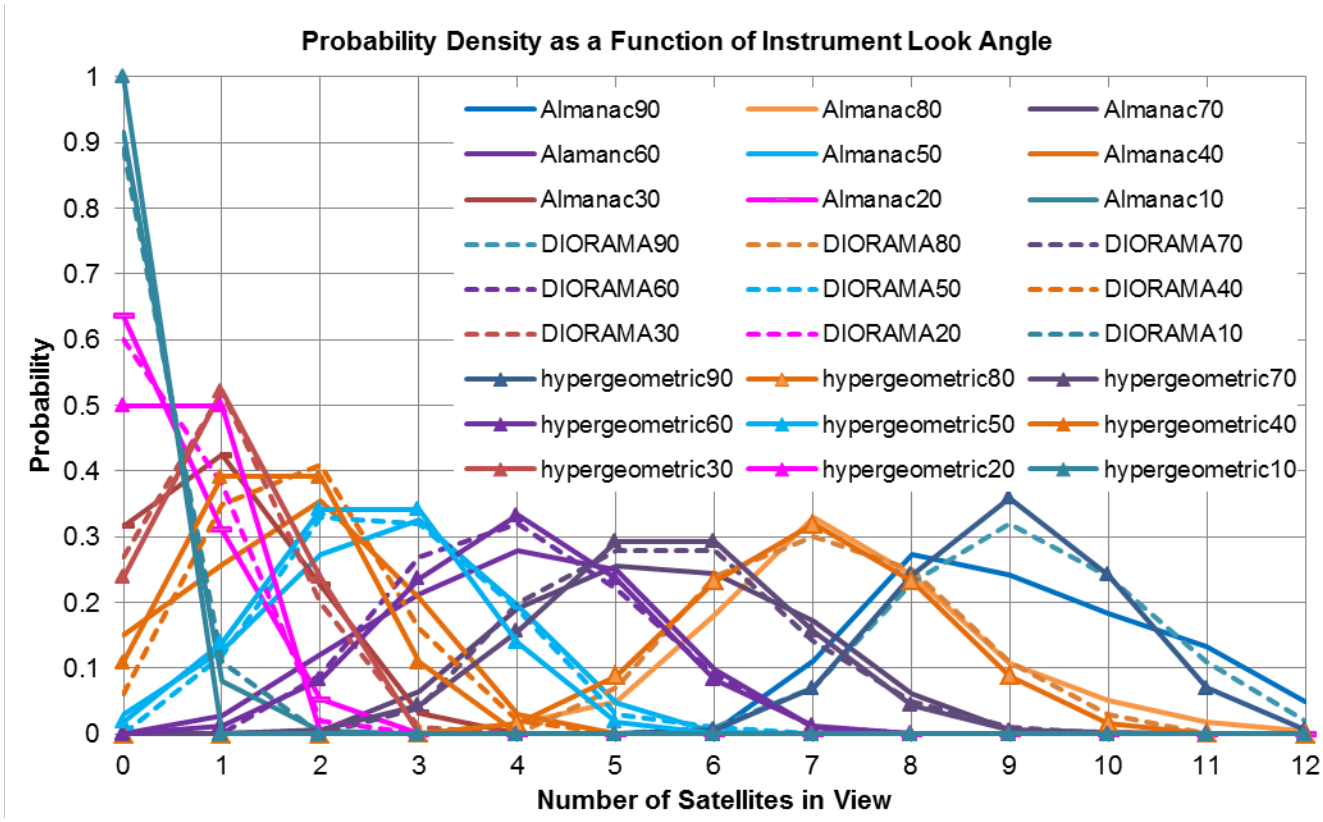
(where Pd_n is the initial Probability of Detection and Pd_{n+1} is the additional Probability of Detection from the next satellite) and is calculated recursively to include all satellites detecting an event in the final value of Probability of Detection.

RESULTS:

The relationship between sensor Maximum Look Angle and average number of satellites in view can be calculated for each assumed sensor Maximum Look Angle using a pure geometric calculation and the Monte Carlo sampled almanac results for the GPS constellation. The results are shown in Figure 2. It can be seen that the assumption of a uniform distribution of satellites around the earth as a representation of the GPS constellation is, in general, a good one, but is better for higher values of sensor maximum look angles. Note, for instance, that the average number of satellites in view suffers at the lower maximum sensor look angles starting at 45 degrees and below. The global average values are good representations for all maximum look angles, but the higher latitudes begin to show degradation in performance for almanac based satellites in view as compared to the pure geometric model. This is caused by the fact that at the North Pole, the actual GPS constellation never rises above about a 45 degree elevation angle – in other words the assumption of uniform satellite distribution in the simple geometric model breaks down over the poles for lower values of sensor maximum look angles. The approximation of uniform coverage is actually reasonable when larger sensor maximum look angles are used because there is then significant overlap between actual satellite orbits overlooking the Earth's surface. From this analysis it is believed that only for a case of considering a specific place on earth, or for very small sensor maximum look angles, would one need to consider applying a correction factor for latitude.

Comparing the probability density functions (PDFs) and cumulative distribution functions (CDFs) in terms of satellites-in-view provides a clear demonstration that coverage provided by GPS-based capabilities is approximately random. Figure 3 compares the PDFs between the random model and almanac data for an instrument having a look-angle between ten and 90 degrees and demonstrates that the two methods show good agreement.

DIORAMA (Distributed Infrastructure Offering Real-time Access to Modeling and Analysis) is a software resource developed by Los Alamos and Sandia National Laboratories for the NNSA that enables, in part, estimation of performance of space-based sensors, including those hosted on GPS satellites. It uses detailed ephemeris associated with the GPS constellation and incorporates a



simple Maximum Look-Angle model for events near the surface of the earth. Between 50 and 90 degrees, results are all within about 10% of each other. Thus, the simple random distribution approximation using a sample size equal to half the total constellation size should be sufficient for estimating coverage associated with GPS-based navigation satellites.

SUMMARY & CONCLUSION

Satellite coverage can be approximated by assuming a uniformly random distribution for both navigation use and nuclear detonation detection capable sensors hosted by GPS. Depending on the capability of the instrument, or nature of signals generated by an event, the approximation can be as good as to within 5%, and generally no worse than within 15% agreement with almanac generated results, except for 20 degrees maximum look angle, which approaches 25%. Overall, the average difference between all three methods for all look angles is about 10%. Estimates from the simple analytic method therefore seem adequate, remarkably across the entire range of possible performance (zero to 90 degrees), for use in informing acquisition decisions and to check performance estimates made with more complex models. When considering performance worldwide, knowing with confidence that a minimum number of satellites will be in-view is likely to be more important than knowing the spatial diversity of cases where there is considerably more than the minimum number in view. In support of the navigation mission, for example, it is better to know the approximate probability of seeing at least four satellites anywhere on the earth than to know which places on the earth have eight, nine, or ten satellites view. For this purpose, a simple parametric random model is particularly useful because it enables estimation of performance and mission risk. Finally, because costs are dominated by the number of fielded spacecraft[2], the method may also be used to estimate the cost and cost risk.

REFERENCES

1. Boelitz, C.A. and Eric V. Beck, "Orbital Analysis Software Index Final Report," PL-TR-95-1139, Rev.1, Vol. II, Phillips Laboratory, Kirtland AFB, July 1997.
2. Global Positioning System: A Comprehensive Assessment of Potential Options and Related Cots is Needed," Report to Congressional Committees, United States Government Accountability Office, GAO-13-729, Sep 2013.

This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.