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Probabilistic Characterization of Partial Volume Effects in Imaging of Rectangular Objects

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1 Introduction

In imaging, a partial volume effect refers to the problem that arises when the system resolution is low relative to the size of the object being imaged [1, 2]. In this setting, it is likely that most voxels occupied by the object are only partially covered, and that the fraction covered in each voxel is low. This makes the problem of object detection and image segmentation very difficult because the algorithms designed for these purposes rely on pixel summary statistics. If the area covered by the object is very low in relatively many of the total number of the voxels the object occupies, these summary statistics may not reach the thresholds required to detect this object. It is thus important to understand the extent of partial volume effect for a given object size and resolution. This technical report focuses on rectangular objects and derives the probability distributions for three quantities for such objects: 1) the number of fully covered voxels, 2) the number of partially covered voxels, and 3) the fractions of the total volume covered in the partially covered voxels. The derivations are first shown for 2-D settings and are then extended to 3-D settings.

2 Two-Dimensional Setting

Assume a rectangular object, such as a sheet, with edge length x along the first dimension and edge length y along the second dimension, randomly centered at location (c_x, c_y) in a voxel. Assume rectangular voxels, each with edge length v_1 along the first dimension and edge length v_2 along the second dimension.

The quantities x, y, v_1 and v_2 are assumed to be known. On the other hand, randomly placing the center of the object in a voxel means that the location (c_x, c_y) is a random variable with the Uniform distribution on a rectangular region $(0, v_1) \times (0, v_2)$. This is equivalent to sampling $c_x \sim \text{Uniform}(0,\ v_1)$ and $c_y \sim \text{Uniform}(0,\ v_2)$ independently from each other (for details on the Uniform distribution, as well as other basic probability concepts discussed throughout this report, see [3]). The object is assumed to fit completely in the field of view,

so the total number of voxels covered by the object is given by $xy/(v_1 v_2)$, with the non-integer part reflecting the fact that some voxels are only partially covered.

We are interested in the following quantities:

1. The number of fully covered voxels, denoted by N_f from here on,
2. The number of partially covered voxels, denoted by N_p from here on,
3. The fractions that are covered in the partially covered voxels.

Randomness in the center location (c_x, c_y) induces randomness in all three of the above quantities of interest. We will obtain the distributions of these random variables.

2.1 1-D Preliminaries

We first obtain the expressions for the variables of interest outlined above in terms of x, y, v_1, v_2, c_x, and c_y.

2.1.1 Number of Fully Covered Voxels

Consider first the number of fully and partially covered voxels in a 1-dimensional setting where a strip of length x along the first dimension is centered at location c_x inside a 1-dimensional interval of length v_1, with $0 \leq c_x \leq v_1$. This is illustrated in Figure 1. Note that if the object length is smaller than the voxel size, i.e., $x < v_1$, the number of fully covered voxels is identically 0, and there is nothing further to consider in that case in terms of the number of fully covered voxels.

Therefore, from here on, we assume that $x \geq v_1$. The number of fully covered voxels in this case is given by

$$f_x = \left(\left\lfloor \frac{c_x + x/2}{v_1} \right\rfloor - \left\lceil \frac{c_x - x/2}{v_1} \right\rceil \right) \tag{1}$$

where $\lfloor n \rfloor$ and $\lceil n \rceil$ denote the floor and the ceiling of a real number n, respectively. Note that f_x is guaranteed to be non-negative when $x \geq v_1$ (this is shown in Appendix D.1), which is our standing assumption in this document, unless noted otherwise. The fully covered voxels are shown in orange in Figure 1.

Let us examine the possible values that f_x can take on. Letting $q_x = c_x/v_1$, we can rewrite (1) as

$$f_x = \left\lfloor q_x + \frac{x/2}{v_1} \right\rfloor - \left\lceil q_x - \frac{x/2}{v_1} \right\rceil = a_x - b_x, \tag{2}$$

with

$$a_x = \left\lfloor q_x + \frac{x/2}{v_1} \right\rfloor \tag{3}$$

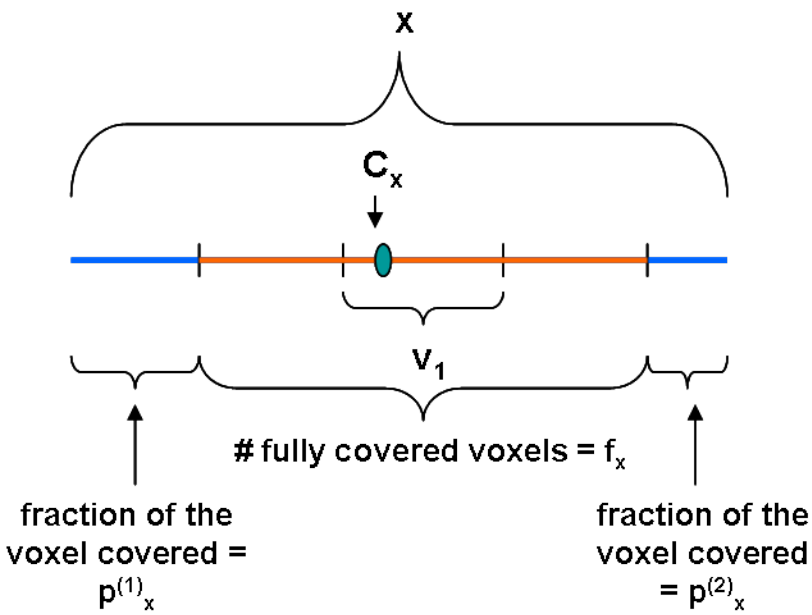

Figure 1: One-dimensional setting. A strip of length x is centered at c_x. The orange part of the strip shows the fully covered voxels, while the blue parts show the partially covered voxels.

and

$$b_x = \left\lceil q_x - \frac{x/2}{v_1} \right\rceil . \tag{4}$$

Since $c_x \sim \text{Uniform}(0, v_1)$, we have $0 \leq c_x \leq v_1$, so $0 \leq q_x \leq 1$. This means that a_x and b_x can each take on two consecutive integer values. At $q_x = 0$, a_x will take on the smaller integer value, which we will denote by k_x, so

$$k_x \equiv \left\lfloor \frac{x/2}{v_1} \right\rfloor . \tag{5}$$

At $q_x = 1$, on the other hand, a_x will take on the larger value of $k_x + 1$.

Now, $a_x = k_x$ as long as

$$q_x + \frac{x/2}{v_1} < k_x + 1 \tag{6}$$

(there is no need for the lower bound because $q_x + \frac{x/2}{v_1} \geq k_x$ by the definition of k_x), implying that

$$q_x < k_x + 1 - \frac{x/2}{v_1} = k_x + 1 - \left(\left\lfloor \frac{x/2}{v_1} \right\rfloor + frac\left(\frac{x/2}{v_1} \right) \right) \tag{7}$$

$$= k_x + 1 - k_x - frac\left(\frac{x/2}{v_1} \right) = 1 - frac\left(\frac{x/2}{v_1} \right), \tag{8}$$

where $frac(n)$ is the fractional part of a real number n, that is,

$$frac(n) = \begin{cases} n - \lfloor n \rfloor & n > 0 \\ \lceil n \rceil - n & n < 0 \end{cases} \tag{9}$$

To simplify the notation, we will define

$$m_x \equiv frac\left(\frac{x/2}{v_1} \right) = \frac{x/2}{v_1} - k_x . \tag{10}$$

Clearly, $0 \leq m_x < 1$. Thus, we have that

$$a_x = \begin{cases} k_x & \text{if } q_x < 1 - m_x \\ k_x + 1 & \text{if } q_x \geq 1 - m_x \end{cases} \tag{11}$$

Now, at $q_x = 0$,

$$b_x = \left\lceil -\frac{x/2}{v_1} \right\rceil = -\left\lfloor \frac{x/2}{v_1} \right\rfloor = -k_x , \tag{12}$$

so b_x is equal to either $-k_x$ or $-k_x + 1$. An analogous calculation to that for a_x shows that

$$b_x = \begin{cases} -k_x & \text{if } q_x < m_x \\ -k_x + 1 & \text{if } q_x \geq m_x \end{cases} \tag{13}$$

Putting (11) and (13) together, we have that

$$f_x = a_x - b_x = \begin{cases} k_x - (-k_x) = 2k_x & \text{if } q_x < min(1-m_x, m_x) \\ k_x - (-k_x+1) = 2k_x - 1 & \text{if } m_x \leq q_x < 1-m_x \\ k_x + 1 - (-k_x) = 2k_x + 1 & \text{if } 1-m_x \leq q_x < m_x \\ k_x + 1 - (-k_x+1) = 2k_x & \text{if } q_x \geq max(1-m_x, m_x) \end{cases} \tag{14}$$

Now,

$$m_x \leq 1 - m_x \tag{15}$$

when

$$m_x < 0.5, \tag{16}$$

so (14) translates to the following:

$$\begin{aligned} &\text{if} \quad m_x < 0.5, \\ &f_x \quad = \begin{cases} 2k_x & \text{if } q_x < m_x \text{ or } \ q_x \geq 1-m_x \\ 2k_x - 1 & \text{if } m_x \leq q_x < 1-m_x \end{cases} \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\text{if} \quad m_x \geq 0.5, \\ &f_x \quad = \begin{cases} 2k_x & \text{if } q_x < 1-m_x \text{ or } \ q_x \geq m_x \\ 2k_x + 1 & \text{if } 1-m_x \leq q_x < m_x \end{cases} \end{aligned} \tag{18}$$

Now, since $c_x \sim \text{Uniform}(0, v_1)$ and $q_x = c_x/v_1$, it follows that $q_x \sim \text{Uniform}(0, 1)$. The probability of q_x being in any subinterval of (0,1) is therefore equal to the length of that subinterval. This means that f_x has the following distribution:

$$\begin{aligned} &\text{if} \quad m_x < 0.5, \\ &f_x \quad = \begin{cases} 2k_x & \text{with probability } 2m_x \\ 2k_x - 1 & \text{with probability } 1 - 2m_x \end{cases} \end{aligned} \tag{19}$$

while

$$\begin{aligned} &\text{if} \quad m_x \geq 0.5, \\ &f_x \quad = \begin{cases} 2k_x & \text{with probability } 2(1-m_x) \\ 2k_x + 1 & \text{with probability } 2m_x - 1 \end{cases} \end{aligned} \tag{20}$$

Thus, f_x always has a two-point distribution that depends on the values of x and v_1 (as these determine whether m_x is below or above 0.5). Note also that the probabilities associated with each of the two points depend only on the absolute distance that m_x is from 0.5, so for example, the probabilities are the same when $m_x = 0.3$ and when $m_x = 0.7$ since these two values are the same distance from 0.5.

It will become convenient in later sections to express (19) and (20) as follows:

$$f_x = \begin{cases} f_x^{(1)} & \text{with probability } p(f_x^{(1)}) \\ f_x^{(2)} & \text{with probability } p(f_x^{(2)}) \end{cases} \tag{21}$$

where

$$\begin{aligned} f_x^{(1)} &= 2k_x, \\ f_x^{(2)} &= \begin{cases} 2k_x - 1 & \text{if } m_x < 0.5 \\ 2k_x + 1 & \text{if } m_x \geq 0.5 \end{cases} \end{aligned} \tag{22}$$

$$\begin{aligned} p(f_x^{(1)}) &= \begin{cases} 2m_x & \text{if } m_x < 0.5 \\ 2(1 - m_x) & \text{if } m_x \geq 0.5 \end{cases} \\ p(f_x^{(2)}) &= 1 - p(f_x^{(1)}). \end{aligned} \tag{23}$$

It can be easily shown (see Appendix A for derivation) that for both ranges of m_x, the expected (mean) value of f_x is given by

$$E(f_x) = \frac{x}{v_1} - 1. \tag{24}$$

Thus, the mean number of fully covered voxels is the total number of covered voxels (with the decimal part reflecting the voxels that are partially covered) minus 1. As derived in Appendix A, the variance of f_x depends on the distance that the value of m_x is from 0.5. Specifically, the variance of f_x is given by

$$Var(f_x) = 4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) \tag{25}$$

for $0 \leq m_x < 1$.

Figure 2 shows the variance as a function of the distance that the value of m_x is from 0.5. At $m_x = 0$ and $m_x = 0.5$, the variance of the number of fully covered voxels is 0 since in those cases the number of fully covered voxels takes on the same value regardless of where the object is centered. On the other hand, the variance is at its maximum value of 0.25 for $m_x = 0.25$ and $m_x = 0.75$ since in those cases, the two values for each of the two-point distributions are equally likely, making the number of fully covered voxels most variable.

The standard deviation of f_x is simply the square root of the variance of f_x, i.e., it is given by

$$\sigma_{f_x} = 2\sqrt{|m_x - 0.5| \cdot (0.5 - |m_x - 0.5|)}. \tag{26}$$

2.1.2 Partially Covered Voxels: Number and Fractions Covered

We can now turn our attention to the partially covered voxels. We still assume that $x \geq v_1$, with a brief discussion devoted to the case when $x < v_1$ at the end of the section. In the 1-dimensional setting, there are 2 partially covered voxels (shown in blue in Figure 1), one on the left of the interval of fully covered voxels, with the fraction of the voxel that is covered equal to

$$p_x^{(1)} = \left\lceil \frac{c_x - x/2}{v_1} \right\rceil - \frac{c_x - x/2}{v_1} = \left\lceil q_x - \frac{x/2}{v_1} \right\rceil - \left(q_x - \frac{x/2}{v_1} \right) \tag{27}$$

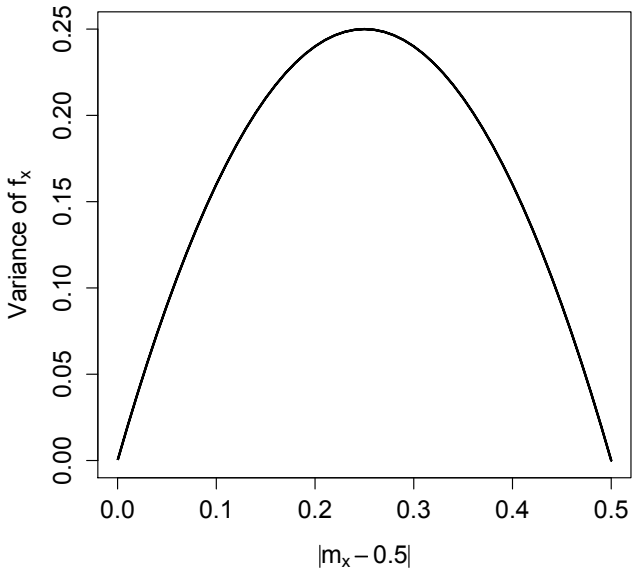

Figure 2: Variance of f_x as a function of $|m_x - 0.5|$, or the distance between m_x and 0.5

and one on the right of the interval of fully covered voxels, with the fraction of the voxel that is covered equal to

$$p_x^{(2)} = \frac{c_x + x/2}{v_1} - \left\lfloor \frac{c_x + x/2}{v_1} \right\rfloor = \left(q_x + \frac{x/2}{v_1} \right) - \left\lfloor q_x + \frac{x/2}{v_1} \right\rfloor . \tag{28}$$

(As a check, the sum $f_x + p_x^{(1)} + p_x^{(2)}$ is equal to x/v_1, the total number of voxels of length v_1 covered by a 1-d strip of length x, with the non-integer part reflecting the fact that some voxels are only partially covered. Note that this implies that once f_x and $p_x^{(1)}$ are known, $p_x^{(2)}$ can be obtained as

$$p_x^{(2)} = x/v_1 - f_x - p_x^{(1)} \tag{29}$$

rather than using the expression in (28)).

Now, since $q_x \sim$ Uniform(0,1), $\left(q_x - \frac{x/2}{v_1} \right)$ is a Uniform random variable on the interval $(-\frac{x/2}{v_1}, 1 - \frac{x/2}{v_1})$, while $\left(q_x + \frac{x/2}{v_1} \right)$ is a Uniform random variable on the interval $(\frac{x/2}{v_1}, 1 + \frac{x/2}{v_1})$. Thus, the length of the interval (the distance between the endpoints) for both random variables is 1. In Appendix E.2.3, it is shown that for for any Uniform random variable R with an integer-valued interval length, the distribution of both $\lceil R \rceil - R$ and $R - \lfloor R \rfloor$ is Uniform(0,1). Thus, both $p_x^{(1)}$ and $p_x^{(2)}$ have a Uniform(0,1) distribution.

Let us briefly examine the number of partially covered voxels in the case when $x < v_1$. In that situation, there are two cases to consider. The first is when the entire object fits into one partially filled voxel. In this case, the fraction covered in the partially filled voxel

is equal to the length of the object scaled to the voxel size, i.e., x/v_1. This happens if two conditions are satisfied: $c_x - x/2 \geq 0$ and $c_x + x/2 \leq v_1$, which translates to the condition that $\frac{x/2}{v_1} \leq q_x \leq 1 - \frac{x/2}{v_1}$. Since $q_x \sim U(0,1)$, this occurs with probability $1 - x/v_1$, or $1 - 2m_x$, because in this case $m_x \equiv \frac{x/2}{v_1}$.

The other subcase of the case when $x < v_1$ is that the object partially covers 2 voxels. Since this occurs if $0 \leq q_x < \frac{x/2}{v_1}$ or if $1 - \frac{x/2}{v_1} < q_x \leq 1$, the probability of this occurring is x/v_1, or $2m_x$. The fractions covered in the two partially covered voxels are still defined as in (27) and (28). However, their distributions are no longer Uniform(0,1) since the fraction covered in either voxel is now at most $x/v_1 < 1$ rather than 1. As shown in Appendix E.3, $p_x^{(1)}$ and $p_x^{(2)}$ is each distributed Uniform(0, x/v_1). The expected value for the fraction covered in each of the 2 voxels is therefore $\frac{x/2}{v_1} = m_x$, so together they are expected to cover x/v_1, as required. The standard deviation for the fraction covered in each of the 2 voxels is $\frac{x/v_1}{\sqrt{12}}$.

Thus, when $x < v_1$, with probability $1 - 2m_x$, there is one partially covered voxel with the fraction covered fixed at x/v_1, and with probability $2m_x$, there are two partially covered voxels with the fractions covered as given in (27) and (28) and distributions, expected values and standard deviations as described in the previous paragraph.

2.1.3 Remaining 1-D Preliminaries

If we consider the analogous 1-dimensional setting along the second dimension with the strip of length y centered at c_y inside a 1-dimensional interval of length v_2, the relevant quantities and results are analogous to those obtained earlier. In particular, if $y < v_2$, $f_y \equiv 0$. Thus, assuming $y \geq v_2$,

$$f_y = \left\lfloor \frac{c_y + y/2}{v_2} \right\rfloor - \left\lceil \frac{c_y - y/2}{v_2} \right\rceil, \tag{30}$$

with the following two-point distribution:

$$\begin{aligned} &\text{if} \quad m_y < 0.5, \\ f_y &= \begin{cases} 2k_y & \text{with probability } 2m_y \\ 2k_y - 1 & \text{with probability } 1 - 2m_y \end{cases} \end{aligned} \tag{31}$$

while

$$\begin{aligned} &\text{if} \quad m_y \geq 0.5, \\ f_y &= \begin{cases} 2k_y & \text{with probability } 2(1 - m_y) \\ 2k_y + 1 & \text{with probability } 2m_y - 1 \end{cases} \end{aligned} \tag{32}$$

where

$$k_y \equiv \left\lfloor \frac{y/2}{v_2} \right\rfloor \tag{33}$$

and

$$m_y \equiv frac\left(\frac{y/2}{v_2}\right) = \frac{y/2}{v_2} - k_y. \tag{34}$$

The distributions in (31) and (32) can be alternatively expressed as follows:

$$f_y = \begin{cases} f_y^{(1)} & \text{with probability } p(f_y^{(1)}) \\ f_y^{(2)} & \text{with probability } p(f_y^{(2)}) \end{cases} \tag{35}$$

where

$$\begin{aligned} f_y^{(1)} &= 2k_y, \\ f_y^{(2)} &= \begin{cases} 2k_y - 1 & \text{if } m_y < 0.5 \\ 2k_y + 1 & \text{if } m_y \geq 0.5 \end{cases} \end{aligned} \tag{36}$$

$$\begin{aligned} p(f_y^{(1)}) &= \begin{cases} 2m_y & \text{if } m_y < 0.5 \\ 2(1 - m_y) & \text{if } m_y \geq 0.5 \end{cases} \\ p(f_y^{(2)}) &= 1 - p(f_y^{(1)}). \end{aligned} \tag{37}$$

Analogously to those of f_x, the expected value and the variance of f_y are given as follows:

$$E(f_y) = \frac{y}{v_2} - 1 \tag{38}$$

and

$$Var(f_y) = 4 \cdot |m_y - 0.5| \cdot (0.5 - |m_y - 0.5|) \tag{39}$$

for $0 \leq m_y < 1$. The standard deviation of f_y is simply the square root of the variance of f_y, i.e., it is given by

$$\sigma_{f_y} = 2\sqrt{|m_y - 0.5| \cdot (0.5 - |m_y - 0.5|)}. \tag{40}$$

In addition, still assuming $y \geq v_2$,

$$p_y^{(1)} = \left\lceil \frac{c_y - y/2}{v_2} \right\rceil - \frac{c_y - y/2}{v_2}, \tag{41}$$

and

$$p_y^{(2)} = \frac{c_y + y/2}{v_2} - \left\lfloor \frac{c_y + y/2}{v_2} \right\rfloor = y/v_2 - f_y - p_y^{(1)}, \tag{42}$$

with each following a Uniform(0,1) distribution, analogously to $p_x^{(1)}$ and $p_x^{(2)}$.

If $y < v_2$, analogously to the derivations in the x dimension, with probability $1 - 2m_y$, there is one partially covered voxel with the fraction covered fixed at y/v_2, and with probability $2m_y$, there are two partially covered voxels with the fractions covered as given in (41) and (42). Each of the two fractions is distributed Uniform(0, y/v_2) and has expected value and standard deviation equal to $\frac{y/2}{v_2} = m_y$ and $\frac{y/v_2}{\sqrt{12}}$, respectively.

2.2 Back to 2-D

We can now derive the number of fully covered voxels N_f, the number of partially covered voxels N_p and the fractions covered in the partially covered voxels in the 2-dimensional setting.

2.2.1 Number of Fully Covered Voxels N_f

In the 2-dimensional setting, the number of fully covered voxels, denoted by N_f, is equal to the product of the numbers of the fully covered voxels for each of the two dimensions. Thus, if $x < v_1$ or $y < v_2$, the number of fully covered voxels is identically equal to 0, leaving nothing further to consider. We will, therefore, assume that both $x \geq v_1$ and $y \geq v_2$ for the remainder of this section, which yields

$$N_f = f_x f_y = \left(\left\lfloor \frac{c_x + x/2}{v_1} \right\rfloor - \left\lceil \frac{c_x - x/2}{v_1} \right\rceil \right) \times \left(\left\lfloor \frac{c_y + y/2}{v_2} \right\rfloor - \left\lceil \frac{c_y - y/2}{v_2} \right\rceil \right), \tag{43}$$

(f_x and f_y are defined in (1) and (30), respectively). These N_f voxels are shown in orange in Figure 3. Since for given values of x, v_1, y, and v_2, f_x and f_y each takes on two possible values, and N_f is the product of f_x and f_y, it follows that N_f takes on 4 possible values that are all possible combinations of f_x and f_y values.

Moreover, f_x and f_y are independent due to the independence of c_x and c_y, so the probabilities associated with each of the 4 possible values of N_f are simply the products of the corresponding probabilities in the 1-dimensional setting. Specifically, the distribution of N_f is given as follows:

$$N_f = \begin{cases} f_x^{(1)} \cdot f_y^{(1)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \\ f_x^{(1)} \cdot f_y^{(2)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \\ f_x^{(2)} \cdot f_y^{(1)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \\ f_x^{(2)} \cdot f_y^{(2)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \end{cases} \tag{44}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22), $f_y^{(1)}$ and $f_y^{(2)}$ as defined in (36), $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23), and $p(f_y^{(1)})$ and $p(f_y^{(2)})$ as defined in (37).

Since $N_f = f_x f_y$ and f_x and f_y are independent, the expected value of N_f in the 2-dimensional setting is given by

$$E(N_f) = E(f_x f_y) = E(f_x)E(f_y) = \left(\frac{x}{v_1} - 1 \right) \cdot \left(\frac{y}{v_2} - 1 \right). \tag{45}$$

The independence of f_x and f_y also implies the independence of f_x^2 and f_y^2. This, put together with (24), (25), (38), and (39), gives that the variance of N_f is equal to

$$\begin{aligned} Var(N_f) &= E(N_f^2) - (E(N_f))^2 = E(f_x^2 f_y^2) - (E(f_x f_y))^2 = E(f_x^2)E(f_y^2) - (E(f_x)E(f_y))^2 \\ &= \left[Var(f_x) + (E(f_x))^2 \right] \cdot \left[Var(f_y) + (E(f_y))^2 \right] - (E(f_x))^2 (E(f_y))^2 \\ &= \left[4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) + \left(\frac{x}{v_1} - 1 \right)^2 \right] \\ &\times \left[4 \cdot |m_y - 0.5| \cdot (0.5 - |m_y - 0.5|) + \left(\frac{y}{v_2} - 1 \right)^2 \right] \\ &- \left(\frac{x}{v_1} - 1 \right)^2 \left(\frac{y}{v_2} - 1 \right)^2 \end{aligned} \tag{46}$$

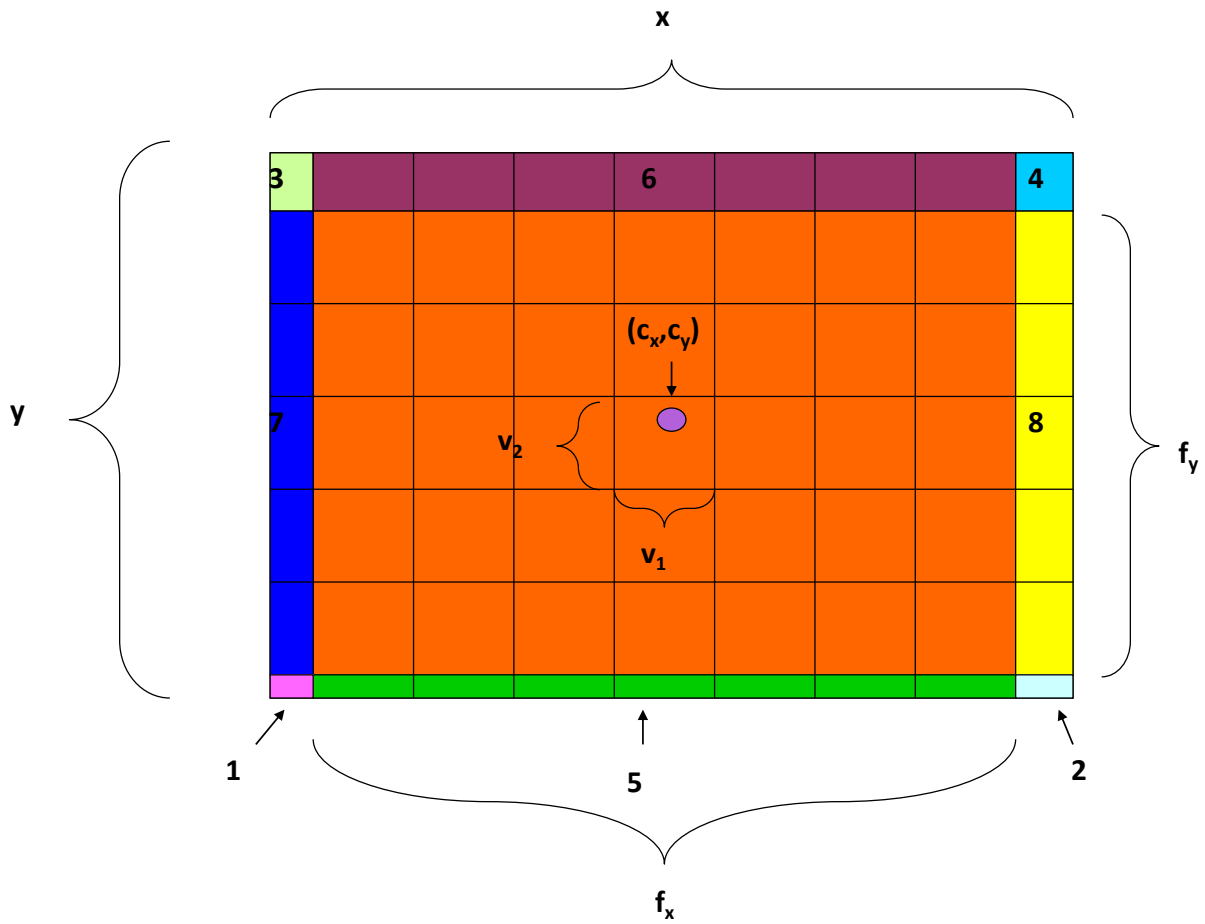

Figure 3: Two-dimensional setting. A rectangular region with side lengths x and y is centered at (c_x, c_y). The voxel horizontal and vertical dimensions are v_1 and v_2, respectively. The orange part of the region consists of fully covered voxels, while the numbered regions consist of partially covered voxels. Also shown are f_x and f_y, the number of voxel lengths fully covered along the x and y dimensions, respectively.

The standard deviation of the number of fully covered voxels is simply the square root of the expression in (46).

2.2.2 Number of Partially Covered Voxels N_p

The remainder of the 2-dimensional object consists of partially covered voxels. In this section, we assume that both $x \geq v_1$ and $y \geq v_2$ (the case when this is not true is covered in detail in Section 2.3). Then the number of partially filled voxels, denoted by N_p, is equal to

$$N_p = 4 + 2(f_x + f_y) \tag{47}$$

as there is one partially filled voxel in each of the 4 corners of the object (regions labeled 1, 2, 3 and 4, in Figure 3), as well as f_x partially covered voxels along each of the 2 horizontal edges of the fully-covered voxel area (regions labeled 5 and 6 in Figure 3), and f_y partially covered voxels along each of the 2 vertical edges of the fully-covered voxel area (regions labeled 7 and 8 in Figure 3).

Since N_p is only a function of f_x and f_y, which are independent of one another, its distribution is given as follows:

$$N_p = \begin{cases} 4 + 2 \cdot (f_x^{(1)} + f_y^{(1)}) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \\ 4 + 2 \cdot (f_x^{(1)} + f_y^{(2)}) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \\ 4 + 2 \cdot (f_x^{(2)} + f_y^{(1)}) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \\ 4 + 2 \cdot (f_x^{(2)} + f_y^{(2)}) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \end{cases} \tag{48}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22), $f_y^{(1)}$ and $f_y^{(2)}$ as defined in (36), $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23), and $p(f_y^{(1)})$ and $p(f_y^{(2)})$ as defined in (37).

Note that the 4 values of N_p in (48) are not distinct. For any set of values of m_x and m_y, two of the four values of N_p above are always identical (which two depends on whether m_x and m_y are below 0.5 or not), so the distribution of N_p reduces to a three-point distribution, with the probabilities for the identical values just adding together.

The expected value of the number of partially filled voxels is given by

$$\begin{aligned} E(N_p) &= E\left(4 + 2(f_x + f_y)\right) = 4 + 2\left(E(f_x) + E(f_y)\right) = 4 + 2\left(\frac{x}{v_1} - 1 + \frac{y}{v_2} - 1\right) \\ &= 2\left(\frac{x}{v_1} + \frac{y}{v_2}\right) \end{aligned} \tag{49}$$

Using the independence of f_x and f_y, we have that the variance of the number of partially filled voxels is given by

$$\begin{aligned} Var(N_p) &= Var\left(4 + 2(f_x + f_y)\right) = 4\left(Var(f_x) + Var(f_y)\right) \\ &= 16 \cdot (|m_x - 0.5| \cdot (0.5 - |m_x - 0.5|)) + 16 \cdot (|m_y - 0.5| \cdot (0.5 - |m_y - 0.5|)) \end{aligned} \tag{50}$$

The standard deviation of N_p is simply the square root of the expression in (50).

2.2.3 Fractions Covered in Partially Covered Voxels

The fraction in each of the N_p partially covered voxels can be summarized by Table 1, which lists the value of the fraction covered, its mean and standard deviation, and the number of voxels with that fraction covered, with its mean and standard deviation, for each region in Figure 3 for reference.

(As a check, note that the sum of the fractions covered listed in Table 1 plus the number of fully covered voxels (given by $f_x f_y$) is equal to

$$\begin{aligned} p_x^{(1)}p_y^{(1)} + p_x^{(2)}p_y^{(1)} + p_x^{(1)}p_y^{(2)} + p_x^{(2)}p_y^{(2)} + f_x p_y^{(1)} + f_x p_y^{(2)} + f_y p_x^{(1)} + f_y p_x^{(2)} + f_x f_y = \\ \left(p_x^{(1)} + p_x^{(2)}\right)\left(p_y^{(1)} + p_y^{(2)}\right) + f_x\left(p_y^{(1)} + p_y^{(2)}\right) + f_y\left(p_x^{(1)} + p_x^{(2)}\right) + f_x f_y = \\ \left(p_x^{(1)} + p_x^{(2)} + f_x\right)\left(p_y^{(1)} + p_y^{(2)} + f_y\right) = (x/v_1)(y/v_2) = \tfrac{xy}{v_1 v_2}, \end{aligned} \tag{51}$$

which is the total number of voxels covered by the object, as required.)

Now, as for the distributions of the fractions covered in the partially covered voxels in Table 1 (bottom 4 values in the 2nd column), we have already derived the distributions of four of these, namely, $p_x^{(1)}$, $p_x^{(2)}$, $p_y^{(1)}$ and $p_y^{(2)}$, and showed them to be Uniform(0,1). The expected (mean) value, the median and the standard deviation of each of these four fractions are therefore 1/2, 1/2, and $1/\sqrt{12} = 0.2887$, respectively. The probability that any of the four of these fractions is between values a and b (with $0 \le a < b \le 1$) is simply the length of that interval, $b - a$. For example, the probability that the fraction of the voxel covered in Region 1 in Figure 3 is between 0.2 and 0.5 is equal to 0.3. The number of voxels at each of these fractions covered is random, equal to either f_x or f_y, as listed in Table 1. The distributions of these two random variables are given in (19)–(20) and (31)–(32).

The remaining four values of the fractions covered that appear in Table 1 (top 4 values in the 2nd column) are two-way products of the first four fractions covered, so they are products of two independent Uniform(0,1) random variables. As shown in Appendix F.2, each of the random variables $p_x^{(1)}p_y^{(1)}$, $p_x^{(2)}p_y^{(1)}$, $p_x^{(1)}p_y^{(2)}$ and $p_x^{(2)}p_y^{(2)}$ has the following probability density function (pdf), denoted by g_W:

$$g_W(w) = -log\ w, \qquad 0 \le w \le 1 \tag{52}$$

and 0 otherwise, with w denoting the particular value of each of the 4 fractions. Thus, to obtain the probability that any of the 4 of these fractions is in a certain interval, one needs to integrate the function in (52) over that interval. Specifically, as shown in Appendix F.2, the probability that any of these four fractions is between values a and b (with $0 \le a < b \le 1$) is equal to

$$P(a \le W \le b) = b(1 - log\ b) - a(1 - log\ a), \tag{53}$$

with $0 \cdot log\ 0 \equiv 0$.

As shown in Appendix F.2, the expectation, the median, and the standard deviation of each of the four fractions $p_x^{(1)}p_y^{(1)}$, $p_x^{(2)}p_y^{(1)}$, $p_x^{(1)}p_y^{(2)}$ and $p_x^{(2)}p_y^{(2)}$ are 1/4, 0.1867, and 0.2205, respectively. Each of these four fractions covered occurs in exactly 1 voxel.

Table 1 gives the expressions for the fractions covered in partially covered voxels within *each* of the 8 regions. However, for a given set of values x, y, v_1 and v_2, one may be more interested in the *entire* distribution of fractions covered in all partially covered voxels without fixing the region. That is, the question of interest may be about the fraction covered by a randomly picked partially covered voxel whatever region it may be in.

As can be surmised from the discussion so far, this distribution is a *random mixture* of the distributions within each of the 8 regions in Table 1, with the mixing fractions equal to the relative frequencies of each of the 8 regions. Note that the relative frequency of each region is the number in the 6th column of Table 1 (i.e., 1 for regions 1 through 4, f_x for regions 5 and 6, and f_y for regions 7 and 8) divided by the total number of partially covered voxels $N_p = 4 + 2(f_x + f_y)$. Thus, the mixing fractions are themselves random, as they are functions of the random variables f_x and f_y. Thus, the probability density function of the fraction covered in any randomly picked partially covered voxel (across all 8 regions) is given by

$$g(F) = \sum_R \sum_{f_x} \sum_{f_y} g(F \mid R, f_x, f_y) \cdot p(R \mid f_x, f_y) \cdot p(f_x, f_y), \tag{54}$$

where

F denotes the fraction covered,

R denotes the region (1 through 8),

$g(F \mid R, f_x, f_y)$ denotes the conditional density of the fraction covered in a randomly picked voxel in a given region R and given values of f_x and f_y,

$p(R \mid f_x, f_y)$ is the relative frequency of the region R for given values of f_x and f_y, and

$p(f_x, f_y)$ is the relative frequency of a given pair of f_x and f_y, which is equal to $p(f_x) \cdot p(f_y)$, as f_x and f_y are independent.

Thus, the pdf of the fraction covered in a randomly picked partially covered voxel is given by

$$g(F) = \sum_R \sum_{f_x} \sum_{f_y} g(F \mid R, f_x, f_y) \cdot p(R \mid f_x, f_y) \cdot p(f_x) \cdot p(f_y) \tag{55}$$

where $p(f_x)$ and $p(f_y)$ are given in (19)–(20) and (31)–(32), respectively, while the expressions for $g(F \mid R, f_x, f_y)$ and $p(R \mid f_x, f_y)$ for each region are given in Table 2. The conditional distributions $g(F \mid R, f_x, f_y)$ given in Table 2 are derived in Appendix B. In particular, for $i = 1, 2$, when $m_x < 0.5$,

$$p_x^{(i)} \mid f_x = 2k_x \sim \text{Uniform}(0, 2m_x), \tag{56}$$

and

$$p_x^{(i)} \mid f_x = 2k_x - 1 \sim \text{Uniform}(2m_x, 1). \tag{57}$$

Relevant Region in Figure 3	Fraction Covered				# Voxels at the Fraction Covered			
	Expression	**Distribution**	**Exp. Value**	**Std. Dev.**	**Expression**	**Distribution**	**Exp. Value**	**Std. Dev.**
1	$p_x^{(1)} p_y^{(1)}$	Eq. (52)	1/4	0.22	1	n/a	1	0
2	$p_x^{(2)} p_y^{(1)}$	Eq. (52)	1/4	0.22	1	n/a	1	0
3	$p_x^{(1)} p_y^{(2)}$	Eq. (52)	1/4	0.22	1	n/a	1	0
4	$p_x^{(2)} p_y^{(2)}$	Eq. (52)	1/4	0.22	1	n/a	1	0
5	$p_y^{(1)}$	U(0,1)	1/2	0.29	f_x	Eq. (19)–(20)	$x/v_1 - 1$	Eq. (26)
6	$p_y^{(2)}$	U(0,1)	1/2	0.29	f_x	Eq. (19)–(20)	$x/v_1 - 1$	Eq. (26)
7	$p_x^{(1)}$	U(0,1)	1/2	0.29	f_y	Eq. (31)–(32)	$y/v_2 - 1$	Eq. (40)
8	$p_x^{(2)}$	U(0,1)	1/2	0.29	f_y	Eq. (31)–(32)	$y/v_2 - 1$	Eq. (40)

Table 1: The fraction of the voxel covered for each of the partially covered voxels, its distribution, expected value and standard deviation, along with the number of voxels with that fraction covered, its distribution, expected value and standard deviation, for each region of Figure 3, for the case when $\boldsymbol{x \geq v_1}$ **and** $\boldsymbol{y \geq v_2}$, i.e., each of the object dimensions is greater or equal to the corresponding voxel dimension.

Region R	$g(F \mid R, f_x, f_y)$	$p(R \mid f_x, f_y)$
1	$g(p_x^{(1)} p_y^{(1)} \mid f_x, f_y)$	$1/(4 + 2(f_x + f_y))$
2	$g(p_x^{(2)} p_y^{(1)} \mid f_x, f_y)$	$1/(4 + 2(f_x + f_y))$
3	$g(p_x^{(1)} p_y^{(2)} \mid f_x, f_y)$	$1/(4 + 2(f_x + f_y))$
4	$g(p_x^{(2)} p_y^{(2)} \mid f_x, f_y)$	$1/(4 + 2(f_x + f_y))$
5	$g(p_y^{(1)} \mid f_y)$	$f_x/(4 + 2(f_x + f_y))$
6	$g(p_y^{(2)} \mid f_y)$	$f_x/(4 + 2(f_x + f_y))$
7	$g(p_x^{(1)} \mid f_x)$	$f_y/(4 + 2(f_x + f_y))$
8	$g(p_x^{(2)} \mid f_x)$	$f_y/(4 + 2(f_x + f_y))$

Table 2: The general expressions for $g(F \mid R, f_x, f_y)$ and $p(R \mid f_x, f_y)$ for each of the 8 regions in the right-hand side of (55). In regions 5 and 6, $g(p_y^{(1)} \mid f_x, f_y)$ and $g(p_y^{(2)} \mid f_x, f_y)$ become $g(p_y^{(1)} \mid f_y)$ and $g(p_y^{(2)} \mid f_y)$, respectively, since $p_y^{(1)}$ and $p_y^{(2)}$ do not depend on f_x. Similarly, in regions 7 and 8, $g(p_x^{(1)} \mid f_x, f_y)$ and $g(p_x^{(2)} \mid f_x, f_y)$ become $g(p_x^{(1)} \mid f_x)$ and $g(p_x^{(2)} \mid f_x)$, respectively, since $p_x^{(1)}$ and $p_x^{(2)}$ do not depend on f_y.

1)	Independently generate f_x according to (19)–(20) and f_y according to (31)–(32) (using Uniform(0,1) random variables, as discussed in Appendix C.2).
2)	Given f_x and f_y generated in step 1, generate a value from the distribution of $p_x^{(1)}$ given f_x and independently a value of $p_y^{(1)}$ given f_y according to (56)–(59).
3)	Set $p_x^{(2)} = x/v_1 - f_x - p_x^{(1)}$ and $p_y^{(2)} = y/v_2 - f_y - p_y^{(1)}$.
4)	Set F to be a sequence of the following $4 + 2(f_x + f_y)$ values: $p_x^{(1)} \cdot p_y^{(1)},\quad p_x^{(1)} \cdot p_y^{(2)},\quad p_x^{(2)} \cdot p_y^{(1)},\quad p_x^{(2)} \cdot p_y^{(2)},\quad \underbrace{p_y^{(1)},\ \ldots,\ p_y^{(1)}}_{f_x \text{ times}},\quad \underbrace{p_x^{(1)},\ \ldots,\ p_x^{(1)}}_{f_y \text{ times}}.$
5)	Repeat steps 1–4 K times for a large value K (e.g., 100000).
	The resulting set of K sequences F will be a sample from the density in (55). This set can then be queried for any summary quantity of interest, such as quantiles, percentiles, means, medians and variances of the fractions covered in all the partially covered voxels.

Table 3: Algorithm for generating values of the fraction covered, F, from the distribution in (55) in the 2-dimensional setting when $x \geq v_1$ and $y \geq v_2$.

When $m_x \geq 0.5$,

$$p_x^{(i)} \mid f_x = 2k_x \sim \text{Uniform}(2m_x - 1, 1) \tag{58}$$

and

$$p_x^{(i)} \mid f_x = 2k_x + 1 \sim \text{Uniform}(0, 2m_x - 1), \tag{59}$$

The distributions of $p_y^{(j)}$ given f_y for $j = 1, 2$ are equivalent to those of $p_x^{(i)}$ given f_x (with x quantities replaced by the equivalent y quantities). Since $p_x^{(1)}$ and $p_x^{(2)}$ given f_x are independent of $p_y^{(1)}$ and $p_y^{(2)}$ given f_y,

$$p_x^{(i)} p_y^{(j)} \mid f_x, f_y = (p_x^{(i)} \mid f_x) \cdot (p_y^{(j)} \mid f_y) \tag{60}$$

for $i, j = 1, 2$. That is, the products $p_x^{(i)} p_y^{(j)}$ given f_x and f_y can be obtained by independently sampling $p_x^{(i)}$ given f_x and $p_y^{(j)}$ given f_y and then multiplying them together.

While the exact expression in for the density of F in (55) can be written down in analytical form, it is quite complex and moreover, it is not very useful in itself. Rather, one needs to know how to sample from this density. Putting together the preceding discussion results in an algorithm for sampling from the density of F in (55), which is summarized in Table 3.

2.3 Object Dimensions Smaller Than Voxel Dimensions

In this section, we consider the situation when at least one of the two object dimensions is smaller than its corresponding voxel dimension, i.e., either $x < v_1$ or $y < v_2$, or both conditions are true. For the remainder of this section, we will refer to any object dimension that is smaller than its corresponding voxel dimension as a "small dimension".

When there is at least one small dimension, all the voxels covered by the object are partially covered. That is, the number of fully covered voxels N_f is identically 0. As for the number of partially covered voxels and their fractions covered, as discussed in Section 2.1, there are two subcases to consider in each dimension: the object could partially cover 1 or 2 voxel lengths in each dimension. Consequently, there are 2 main cases to consider, each with its own set of subcases:

1. 1 small dimension, i.e., one of the following is true:

   $x \geq v_1$, $y < v_2$

   $x < v_1$, $y \geq v_2$

   with each of the two possibilities having

   (a) 1 voxel length is partially covered along the small dimension (illustrated in the left-hand plot of Figure 4) or

   (b) 2 voxel lengths are partially covered along the small dimension (illustrated in the right-hand plot of Figure 4)

2. 2 small dimensions, i.e., $x < v_1$, $y < v_2$

   (a) 1 voxel length is partially covered along the x dimension and 1 voxel length is partially covered along the y dimension (illustrated in the left-hand plot of Figure 5)

   (b) 1 voxel length is partially covered along the x dimension and 2 voxel lengths are partially covered along the y dimension (illustrated in the center plot of Figure 5)

   (c) 2 voxel lengths are partially covered along the x dimension and 1 voxel length is partially covered along the y dimension (illustrated in the center plot of Figure 5)

   (d) 2 voxel lengths are partially covered along the x dimension and 2 voxel lengths are partially covered along the y dimension (illustrated in the right-hand plot of Figure 5)

While there is no randomness associated with which of the 2 groups of cases (or which specific case within a group) will apply (as the cases are defined entirely by the voxel and object dimensions, which are assumed to be known), within each case, there is randomness associated with which of the subcases will occur (note that the probabilities of all the subcases within each case must add up to 1).

2.3.1 Number of Partially Filled Voxels N_p

1. 1 small dimension, i.e., one of the following is true:

   $x \geq v_1$ and $y < v_2$

   $x < v_1$ and $y \geq v_2$

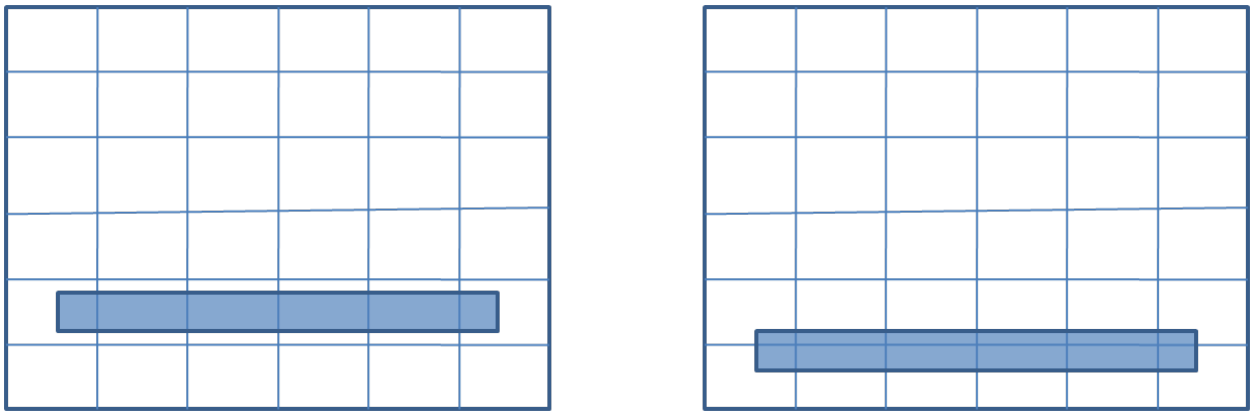

Figure 4: Left-hand plot: The object (shaded blue) on a grid of voxels in the case of 1 small dimension with the object fitting entirely within one voxel length along the small dimension (cases 1(a) and 2(a) of Section 2.3). Right-hand plot: The object (shaded blue) on a grid of voxels in the case of 1 small dimension with the object covering parts of two voxel lengths along the small dimension (cases 1(b) and 2(b) of Section 2.3). Note that, as pictured, the y dimension is assumed to be the small dimension.

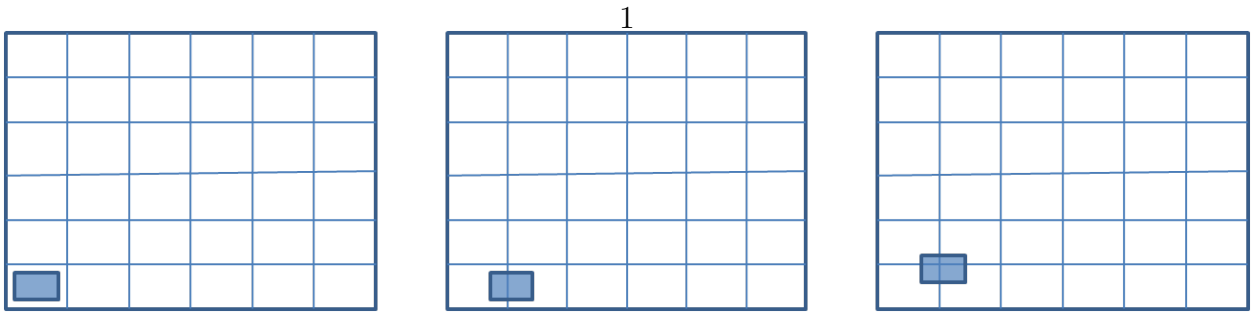

Figure 5: Left-hand plot: The object (shaded blue) on a grid of voxels in the case of 2 small dimensions, with the object fitting entirely within one voxel length along each of the two dimensions (case 3(a) of Section 2.3). Center plot: The object (shaded blue) on a grid of voxels in the case of 2 small dimensions, with the object fitting entirely within one voxel length along one of the dimensions, but covering parts of two voxel lengths along the other small dimension (cases 3(b) and 3(c) of Section 2.3). Right-hand plot: The object (shaded blue) on a grid of voxels in the case of 2 small dimensions, with the object covering parts of two voxel lengths along each of the two dimensions (case 3(d) of Section 2.3).

Since the two dimensions are interchangeable, let us denote the small dimension with y and the other dimension with x (i.e., x and y need not correspond to horizontal and vertical dimensions, respectively). The voxel dimensions corresponding to x and y will be denoted as v_1 and v_2, respectively, as before. Thus, $x \geq v_1$ and $y < v_2$. As shown in Appendix D.2, the number of partially covered voxels in this case is given by

$$N_p = \begin{cases} f_x + 2 & \text{if } m_y \leq q_y \leq 1 - m_y \\ 2(f_x + 2) & \text{otherwise} \end{cases} \tag{61}$$

The distribution of N_p (also derived in Appendix D.2) is given by

$$N_p = \begin{cases} f_x^{(1)} + 2 & \text{with probability } p(f_x^{(1)}) \cdot (1 - y/v_2) \\ f_x^{(2)} + 2 & \text{with probability } p(f_x^{(2)}) \cdot (1 - y/v_2) \\ 2(f_x^{(1)} + 2) & \text{with probability } p(f_x^{(1)}) \cdot (y/v_2) \\ 2(f_x^{(2)} + 2) & \text{with probability } p(f_x^{(2)}) \cdot (y/v_2) \end{cases} \tag{62}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22), and $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23).

As shown in Appendix D.2, the expected value of N_p is given by

$$E(N_p) = \left(\frac{x}{v_1} + 1\right)\left(\frac{y}{v_2} + 1\right). \tag{63}$$

The variance of N_p is given by

$$Var(N_p) = 4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) \cdot \left(1 + \frac{3y}{v_2}\right) + \frac{y}{v_2}\left(1 - \frac{y}{v_2}\right)\left(\frac{x}{v_1} + 1\right)^2 \tag{64}$$

The standard deviation of N_p is the square root of the expression in (64).

2. 2 small dimensions: $x < v_1$ and $y < v_2$
   As shown in Appendix D.2, the number of partially covered voxels is given by

$$N_p = \begin{cases} 1 & \text{if } m_x \leq q_x \leq 1 - m_x \text{ and } m_y \leq q_y \leq 1 - m_y \\ 2 & \text{if } m_x \leq q_x \leq 1 - m_x \text{ and } (0 \leq q_y < m_y \text{ or } 1 - m_y < q_y \leq 1) \\ 2 & \text{if } m_y \leq q_y \leq 1 - m_y \text{ and } (0 \leq q_x < m_x \text{ or } 1 - m_x < q_x \leq 1) \\ 4 & \text{if } (0 \leq q_x < m_x \text{ or } 1 - m_x < q_x \leq 1) \text{ and } (0 \leq q_y < m_y \text{ or } 1 - m_y < q_y \leq 1) \end{cases} \tag{65}$$

   The distribution of N_p is thus given by

$$N_p = \begin{cases} 1 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\left(1 - \frac{y}{v_2}\right) \\ 2 & \text{with probability } \frac{x}{v_1}\left(1 - \frac{y}{v_2}\right) + \frac{y}{v_2}\left(1 - \frac{x}{v_1}\right) \\ 4 & \text{with probability } \frac{x}{v_1} \cdot \frac{y}{v_2} \end{cases} \tag{66}$$

As shown in Appendix D.2, the expected value of N_p is given by

$$E(N_p) = \left(\frac{x}{v_1} + 1\right)\left(\frac{y}{v_2} + 1\right) \tag{67}$$

and the variance of N_p is given by

$$Var(N_p) = \left(\frac{3x}{v_1} + 1\right)\left(\frac{3y}{v_2} + 1\right) - \left(\frac{x}{v_1} + 1\right)^2\left(\frac{y}{v_2} + 1\right)^2. \tag{68}$$

Note that the the expected number of partially filled voxels is the same in all three cases.

2.3.2 Fractions Covered in Partially Covered Voxels

1. 1 small dimension, i.e, one of the following is true:

   $x \geq v_1$ and $y < v_2$

   $x < v_1$ and $y \geq v_2$

   Again, we will denote the small dimension with y and the other dimension with x since the two dimensions are interchangeable. Thus, $x \geq v_1$ and $y < v_2$. As discussed in Section 2.3.1, in this case there are either $f_x + 2$ or $2(f_x + 2)$ partially covered voxels. The former subcase, referred to as 1(a) at the beginning of Section 2.3, corresponds to just 1 partially covered voxel length along the y dimension (shown in the left-hand plot of Figure 4). In this case, which occurs with probability $1-y/v_2$ (as shown in Appendix D.2), the fraction covered in the f_x voxels is y/v_2 each, while the fraction covered in the 2 remaining voxels is $p_x^{(1)}(y/v_2)$ and $p_x^{(2)}(y/v_2)$ (this is easiest to obtain by examining Figure 4). Since $x \geq v_1$, each of $p_x^{(1)}$ and $p_x^{(2)}$ has a Uniform(0,1) distribution. Thus, $p_x^{(1)}(y/v_2)$ and $p_x^{(2)}(y/v_2)$ each has a Uniform(0, y/v_2) distribution, so the probability that any one of these two fractions is between two values a and b with $0 \leq a < b \leq y/v_2$ is equal to $(b-a)v_2/y$. The expected value, median, and standard deviation of each of these two fractions are $y/(2v_2)$, $y/(2v_2)$, and $y/(\sqrt{12}v_2)$, respectively.

   The latter subcase, 1(b), corresponds to 2 partially covered voxel lengths along the y dimension (as shown in the right-hand plot of Figure 4), one $p_y^{(1)}$ and the other $p_y^{(2)}$ long. In this case, which occurs with probability y/v_2 (as shown in Appendix D.2), there are f_x voxels with the fraction covered equal to $p_y^{(1)}$, f_x voxels with the fraction covered equal to $p_y^{(2)}$ voxels, 1 voxel with the fraction covered equal to $p_x^{(1)}p_y^{(1)}$, 1 voxel with the fraction covered equal to $p_x^{(1)}p_y^{(2)}$, 1 voxel with the fraction covered equal to $p_x^{(2)}p_y^{(1)}$, and 1 voxel with the fraction covered equal to $p_x^{(2)}p_y^{(2)}$. As discussed at the end of Section 2.1.3, $p_y^{(1)}$ and $p_y^{(2)}$ each has a Uniform(0, y/v_2) distribution, so the probability that any one of these two fractions is between two values a and b with $0 \leq a < b \leq y/v_2$ is equal to $(b-a)v_2/y$. Each fraction's expected value, median and standard deviation are $y/(2v_2)$, $y/(2v_2)$, and $y/(\sqrt{12}v_2)$, respectively.

Since $p_x^{(1)}$ and $p_x^{(2)}$ each has a Uniform(0,1) distribution, and they are independent from $p_y^{(1)}$ and $p_y^{(2)}$, according to the results obtained in Appendix F.2, the product of any one of $p_x^{(1)}$ or $p_x^{(2)}$ with any one of $p_y^{(1)}$ or $p_y^{(2)}$ has a probability density function given by

$$g_W(w) = \frac{v_2}{y} \cdot (log(y/v_2) - log\ w)\,, \quad 0 \leq w \leq y/v_2 \tag{69}$$

and 0 otherwise. The probability that any one of these products is in the interval (A, B) with $0 \leq A < B \leq y/v_2$ is therefore equal to

$$P(A \leq W \leq B) = \frac{v_2}{y} \cdot (log(y/v_2) + 1) \cdot (B - A) \ + \ \frac{v_2}{y} \cdot (A\ log\ A - B\ log\ B) \tag{70}$$

with $0 \cdot log\ 0 \equiv 0$. According to the results obtained in Appendix F.2, the expected value, the median, and standard deviation of each of these 4 products is $y/(4v_2)$, $(0.19y)/v_2$, and $(0.22y)/v_2$, respectively. Table 4 summarizes the fractions covered and the corresponding number of voxels at that fraction for each of the two subcases of the case of 1 small dimension in the 2-dimensional setting. The small dimension is denoted with y (i.e., $x \geq v_1$ and $y < v_2$).

As discussed in Section 2.2.3, for a given set of values x, y, v_1 and v_2, one may be more interested in the *entire* distribution of fractions covered in all partially covered voxels without fixing the region. To obtain this distribution, we follow the steps equivalent to those outlined in Section 2.2.3 and adapt them to the case of 1 small dimension.

As Table 4 shows, there are 2 subcases, one with 3 and the other with 6 regions. In the first subcase (labeled (a) in the table), the relative frequency of each region is the number in the 8th column of Table 4, (i.e., f_x for region 1 and 1 for regions 2 and 3) divided by the total number of partially covered voxels $N_p = f_x + 2$. In the second subcase (labeled (b) in the table), the relative frequency of each region is the number in the 8th column of Table 1, (i.e., f_x for regions 1 and 2 and 1 for regions 4 through 6) divided by the total number of partially covered voxels $N_p = 2(f_x + 2)$.

These relative frequencies, or mixing fractions, are themselves random, as was the case in Section 2.2.3, as they are functions of f_x. Thus, the probability density function of the fraction covered F in any randomly picked partially covered voxel (across both subcases and all regions within each subcase) is given by

$$g(F) = \sum_R \sum_S \sum_{f_x} g(F \mid R, S, f_x) \cdot p(R \mid S, f_x) \cdot p(S \mid f_x) \cdot p(f_x), \tag{71}$$

where

R denotes the region (1 through 3 for $S = (a)$ and 1 through 6 for $S = (b)$),

S denotes the subcase (a) or (b) in Table 4,

$g(F \mid R, S, f_x)$ denotes the conditional density of the fraction covered in a randomly picked voxel in a given region R within the given subcase S and the value of f_x,

$p(R \mid S, f_x)$ is the relative frequency of the region R for a given subcase S and value of f_x,

$p(S \mid f_x)$ denotes the conditional probability (relative frequency) of each subcase given the value of f_x and is equal to the unconditional probability $p(S)$ as the probability of either subcase does not depend on f_x, as can be seen from Table 4,

$p(f_x)$ is the probability (relative frequency) of the value of f_x.

Thus, the pdf of the fraction covered in a randomly picked partially covered voxel is given by

$$g(F) = \sum_R \sum_S \sum_{f_x} g(F \mid R, S, f_x) \cdot p(R \mid S, f_x) \cdot p(S) \cdot p(f_x), \tag{72}$$

where $p(f_x)$ is as given in (19)–(20), $p(S)$ is given by

$$p(S) = \begin{cases} 1 - \frac{y}{v_2} & S = (a) \\ \frac{y}{v_2} & S = (b) \end{cases} \tag{73}$$

and the expressions for $g(F \mid R, S, f_x)$ and $p(R \mid S, f_x)$ for each subcase and region are given in Table 5.

The conditional distributions $g(F \mid R, S, f_x)$ given in Table 5 can be easily obtained from the results derived in Appendix B. Since, as shown in Appendix B, $p_x^{(i)} \mid f_x$, $i = 1, 2$ are all Uniform random variables, as specified in (56)–(59), and y/v_2 is a constant, $p_x^{(i)} \cdot (y/v_2)$ can be obtained by first generating a random variable from the appropriate distribution in (56)–(59) and multiplying it by y/v_2. As discussed at the end of Section 2.1.3, the distribution of each of $p_y^{(1)}$ and $p_y^{(2)}$ is Uniform$(0, y/v_2)$. Since $p_y^{(1)}$ and $p_y^{(2)}$ are independent of $p_x^{(1)}$ and $p_x^{(2)}$ given f_x,

$$p_x^{(i)} p_y^{(j)} \mid f_x = (p_x^{(i)} \mid f_x) \cdot p_y^{(j)} \tag{74}$$

for $i, j = 1, 2$. That is, the products $p_x^{(i)} p_y^{(j)}$ given f_x can be obtained by independently sampling $p_x^{(i)}$ given f_x and $p_y^{(j)}$ and then multiplying them together.

Putting together the preceding discussion results in an algorithm for sampling from the density of F in (72), which is summarized in Table 6.

2. 2 small dimensions, i.e., $x < v_1$, $y < v_2$

   As discussed in Section 2.3.1, in this case there are either 1, 2, or 4 partially covered voxels. These subcases are illustrated in the left-hand, middle, and right-hand plots of Figure 5. By referring to these plots, it is easily seen that when there is 1 partially covered voxel, the fraction covered by it is the entire object area scaled to the voxel area, i.e., $xy/(v_1 v_2)$. As shown in Appendix D.2, this occurs with probability $(1 - x/v_1)(1 - y/v_2)$.

Sub-case	Proba-bility	Reg-ion	Fraction Covered				# Voxels at the Fraction Covered			
			Expres-sion	Distri-bution	Exp. Value	Std. Dev.	Expres-sion	Distri-bution	Exp. Value	Std. Dev.
(a)	$1-\frac{y}{v_2}$	1	$\frac{y}{v_2}$	n/a	$\frac{y}{v_2}$	0	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		2	$p_x^{(1)}\frac{y}{v_2}$	$\mathrm{U}(0,\frac{y}{v_2})$	$\frac{y}{2v_2}$	$\frac{y}{\sqrt{12}v_2}$	1	n/a	1	0
		3	$p_x^{(2)}\frac{y}{v_2}$	$\mathrm{U}(0,\frac{y}{v_2})$	$\frac{y}{2v_2}$	$\frac{y}{\sqrt{12}v_2}$	1	n/a	1	0
(b)	$\frac{y}{v_2}$	1	$p_y^{(1)}$	$\mathrm{U}(0,\frac{y}{v_2})$	$\frac{y}{2v_2}$	$\frac{y}{\sqrt{12}v_2}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		2	$p_y^{(2)}$	$\mathrm{U}(0,\frac{y}{v_2})$	$\frac{y}{2v_2}$	$\frac{y}{\sqrt{12}v_2}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		3	$p_x^{(1)}p_y^{(1)}$	Eq.(69)	$\frac{y}{4v_2}$	$\frac{0.22y}{v_2}$	1	n/a	1	0
		4	$p_x^{(1)}p_y^{(2)}$	Eq.(69)	$\frac{y}{4v_2}$	$\frac{0.22y}{v_2}$	1	n/a	1	0
		5	$p_x^{(2)}p_y^{(1)}$	Eq.(69)	$\frac{y}{4v_2}$	$\frac{0.22y}{v_2}$	1	n/a	1	0
		6	$p_x^{(2)}p_y^{(2)}$	Eq.(69)	$\frac{y}{4v_2}$	$\frac{0.22y}{v_2}$	1	n/a	1	0

Table 4: The fraction of the voxel covered for each of the partially covered voxels, its distribution, expected value and standard deviation, along with the number of voxels with that fraction covered, its distribution, expected value and standard deviation, for the case of 1 small dimension in the two-dimensional setting. The y quantities correspond to the small dimension, so $x \geq v_1$ and $y < v_2$.

Subcase S	Region R	$g(F \mid R, S, f_x)$	$p(R \mid S, f_x)$
(a)	1	constant at $\frac{y}{v_2}$	$f_x/(2+f_x)$
	2	$g(p_x^{(1)}\frac{y}{v_2} \mid f_x)$	$1/(2+f_x)$
	3	$g(p_x^{(2)}\frac{y}{v_2} \mid f_x)$	$1/(2+f_x)$
(b)	1	$g(p_y^{(1)})$	$f_x/(4+2f_x)$
	2	$g(p_y^{(2)})$	$f_x/(4+2f_x)$
	3	$g(p_x^{(1)}p_y^{(1)} \mid f_x)$	$1/(4+2f_x)$
	4	$g(p_x^{(1)}p_y^{(2)} \mid f_x)$	$1/(4+2f_x)$
	5	$g(p_x^{(2)}p_y^{(1)} \mid f_x)$	$1/(4+2f_x)$
	6	$g(p_x^{(2)}p_y^{(2)} \mid f_x)$	$1/(4+2f_x)$

Table 5: The general expressions for $g(F \mid R, S, f_x)$ and $p(R \mid S, f_x)$ in the right-hand side of (72). In regions 1 and 2 of subcase (b), $g(p_y^{(1)} \mid f_x)$ and $g(p_y^{(2)} \mid f_x)$ become $g(p_y^{(1)})$ and $g(p_y^{(2)})$, respectively, since $p_y^{(1)}$ and $p_y^{(2)}$ do not depend on f_x.

1) Generate f_x according to (19)–(20) (using Uniform(0,1) random variables, as discussed in Appendix C.2).

2) Given f_x generated in step 1, generate $p_x^{(1)}$ from the appropriate distribution specified in (56)–(59).

3) Set $p_x^{(2)} = x/v_1 - f_x - p_x^{(1)}$

4) Generate the subcase $S =$ (a) or (b) according to their probabilities in Table 4 (using Uniform (0,1) random variables, as discussed in Appendix C.2).

5) If $S = (a)$ in step 4,
   – set F to be a sequence of the following $f_x + 2$ values:

$$\underbrace{y/v_2, \quad ..., y/v_2}_{f_x \text{ times}}, \quad (y/v_2) \cdot p_x^{(1)}, \quad (y/v_2) \cdot p_x^{(2)}.$$

   If $S = (b)$ in step 4,
   – generate $p_y^{(1)} \sim U(0, y/v_2)$,
   – set $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and
   – set F to be a sequence of the following $2f_x + 4$ values:

$$\underbrace{p_y^{(1)}, \quad ..., p_y^{(1)}}_{f_x \text{ times}}, \quad \underbrace{p_y^{(2)}, \quad ..., p_y^{(2)}}_{f_x \text{ times}}, \quad p_x^{(1)} \cdot p_y^{(1)}, \quad p_x^{(1)} \cdot p_y^{(2)}, \quad p_x^{(2)} \cdot p_y^{(1)}, \quad p_x^{(2)} \cdot p_y^{(2)}.$$

6) Repeat steps 1–5 K times for a large value K (e.g., 100000).

   The resulting set of K sequences F will be a sample from the density in (72). This set can then be queried for any summary quantity of interest, such as quantiles, percentiles, means, medians and variances of the fractions covered in all the partially covered voxels.

Table 6: Algorithm for generating values of the fraction covered, F, from the distribution in (72) in the 2-dimensional setting with 1 small dimension (y is assumed to be the small dimension, so $x \geq v_1$ and $y < v_2$).

Two partially covered voxels occur when there is 1 partially covered voxel length covered along one dimension and 2 partially covered voxel lenghts along the other. When the former is along the x dimension and the latter is along the y dimension (this occurs with probability $(1 - x/v_1)y/v_2$, as shown in Appendix D.2), the fractions covered in the two voxels are $p_y^{(1)}(x/v_1)$ and $p_y^{(2)}(x/v_1)$, respectively. Since $p_y^{(1)}$ and $p_y^{(2)}$ are each Uniform(0, y/v_2), each of the two fractions covered each has Uniform(0, $\frac{xy}{v_1 v_2}$) distribution, and expected value and standard deviation of $\frac{xy}{2v_1v_2}$ and $\frac{xy}{\sqrt{12}v_1v_2}$, respectively. The probability of any one of the two fractions being between two values a and b with $0 \leq a < b \leq xy/(v_1v_2)$ is $(b-a)\frac{v1v2}{xy}$.

On the other hand, when there are 2 partially covered voxel lengths along x dimension and 1 partially covered voxel length along y dimension (this occurs with probability $x/v_1(1 - y/v_2)$, as shown in Appendix D.2), the fractions covered are $p_x^{(1)}(y/v_2)$ and $p_x^{(2)}(y/v_x)$, respectively. Since $p_x^{(1)}$ and $p_x^{(2)}$ are each Uniform(0, x/v_1), each of the two fractions covered each has Uniform(0, $\frac{xy}{v_1v_2}$) distribution, and expected value and standard deviation of $\frac{xy}{2v_1v_2}$ and $\frac{xy}{\sqrt{12}v_1v_2}$, respectively. The probability of any one of the two fractions being between two values a and b with $0 \leq a < b \leq xy/(v_1v_2)$ is $(b-a)\frac{v1v2}{xy}$.

Finally, four partially covered voxels occur when there are 2 partially covered voxel lenghts covered along both dimensions. This occurs with probability $xy/(v_1v_2$. In that case, the fractions covered in the four voxels are $p_x^{(1)}p_y^{(1)}$, $p_x^{(1)}p_y^{(2)}$, $p_x^{(2)}p_y^{(1)}$, and $p_x^{(2)}p_y^{(2)}$. Since each of $p_x^{(1)}$ and $p_x^{(2)}$ has a Uniform(0, x/v_1) distribution and each of $p_y^{(1)}$ and $p_y^{(2)}$ has a Uniform(0, y/v_2) distribution (and the first two are independent of the other two), according to the results derived in Appendix F.2, the probability density function for each of the four products is given by

$$g_W(w) = \frac{v_1v_2}{xy} \cdot \left(log\left(\frac{xy}{v_1v_2}\right) - log\ w \right), \quad 0 \leq w \leq \frac{xy}{v_1v_2} \tag{75}$$

and 0 otherwise. The probability that any one of these products is in the interval (A, B) with $0 \leq A < B \leq xy/(v_1v_2)$ is therefore equal to

$$P(A \leq W \leq B) = \frac{v_1v_2}{xy} \cdot \left(log\left(\frac{xy}{v_1v_2}\right) + 1 \right) \cdot (B - A) + \frac{v_1v_2}{xy} \cdot (A\ log\ A - B\ log\ B). \tag{76}$$

with $0 \cdot log\ 0 \equiv 0$. The expected value, median, and the standard deviation of each of the fractions is equal to $xy/(4v_1v_2)$, $(0.19xy)/(v_1v_2)$, and $(0.22xy)/(v_1v_2)$, respectively.

Accordingly, Table 7 summarizes the fractions covered and the corresponding number of voxels at that fraction for each of the four subcases of the case of 2 small dimensions in the 2-dimensional setting. To obtain the distribution of the fraction covered across all subcases and regions, we follow the steps equivalent to those outlined in Section 2.2.3 and adapt them to the case of 2 small dimensions.

As Table 7 shows, there are 4 subcases. In the first subcase (labeled (a) in the table), there is only 1 region, so it has the the relative frequency of of 1. In the second and

third subcases (labeled (b) and (c) in the table), there are 2 equally likely regions, so each one has a relative frequency of 1/2 within each subcase. In the fourth subcase (labeled (d) in the table), there are 4 equally likely regions, so each region has a relative frequency of 1/4. Note that the relative frequency of the region is fixed in all 4 subcases.

Thus, the probability density function of the fraction covered F in any randomly picked partially covered voxel (across the four subcases and all regions within each subcase) is given by

$$g(F) = \sum_R \sum_S g(F \mid R, S) \cdot p(R \mid S) \cdot p(S), \tag{77}$$

where, as before,

- R denotes the region (1 for $S = (a)$, 1 or 2 for $S =$ (b) or (c) and and 1 through 4 for $S = (d)$),
- S denotes the subcase (a), (b), (c) or (d) in Table 7,
- $g(F \mid R, S)$ denotes the conditional density of the fraction covered in a randomly picked voxel in a given region R within the given subcase S,
- $p(R \mid S)$ is the relative frequency of the region R for a given subcase S
- $p(S)$ denotes the probability (relative frequency) of each subcase and is given in Table 7.

and the expressions for $g(F \mid R, S)$ and $p(R \mid S)$ for each subcase and region are given in Table 8. The distributions $g(F \mid R, S)$ given in Table 8 can be easily obtained from the results derived earlier in this section and in Section 2.1. Recall in particular that

$$p_x^{(i)} \sim U(0, x/v_1) \tag{78}$$

for $i = 1, 2$ and

$$p_y^{(j)} \sim U(0, y/v_2) \tag{79}$$

for $j = 1, 2$. Furthermore, $p_x^{(i)}$ and $p_y^{(j)}$ are independent from each other for $i, j = 1, 2$, implying that the distributions of the products can be generated by independently sampling $p_x^{(i)}$ and $p_y^{(j)}$ and then multiplying them together.

Putting together the preceding discussion results in an algorithm for sampling from the density of F in (77), which is summarized in Table 9.

2.4 Summary

Table 10 contains the summary of the resulting variables of interest when a rectangular 2-dimensional object with edge lengths x and y is placed inside a rectangular grid of voxels, each with edge lengths v_1 and v_2, with $x \geq v_1$ and $y \geq v_2$ and with the center of the object randomly located inside a voxel at a point (c_x, c_y) (with $0 \leq c_x \leq v_1$ and $0 \leq c_y \leq v_2$). Tables 11 and 12 contain the summaries equivalent to that in Table 10 for the cases of 1 small dimension (either $x < v_1$ or $y < v_2$, but not both) and 2 small dimensions ($x < v_1$ and $y < v_2$), respectively.

Sub-case	Proba-bility	Reg-ion	Fraction Covered				# Voxels at the Fraction Covered
			Expres-sion	Distri-bution	Exp. Value	Std. Dev.	
(a)	$\left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)$	1	$\frac{xy}{v_1 v_2}$	n/a	$\frac{xy}{v_1 v_2}$	0	1
(b)	$\left(1-\frac{x}{v_1}\right)\frac{y}{v_2}$	1	$p_y^{(1)}(x/v_1)$	$\mathrm{U}\left(0, \frac{xy}{v_1 v_2}\right)$	$\frac{xy}{2v_1 v_2}$	$\frac{xy}{\sqrt{12}v_1 v_2}$	1
		2	$p_y^{(2)}(x/v_1)$	$\mathrm{U}\left(0, \frac{xy}{v_1 v_2}\right)$	$\frac{xy}{2v_1 v_2}$	$\frac{xy}{\sqrt{12}v_1 v_2}$	1
(c)	$\frac{x}{v_1}\left(1-\frac{y}{v_2}\right)$	1	$p_x^{(1)}(y/v_2)$	$\mathrm{U}\left(0, \frac{xy}{v_1 v_2}\right)$	$\frac{xy}{2v_1 v_2}$	$\frac{xy}{\sqrt{12}v_1 v_2}$	1
		2	$p_x^{(2)}(y/v_2)$	$\mathrm{U}\left(0, \frac{xy}{v_1 v_2}\right)$	$\frac{xy}{2v_1 v_2}$	$\frac{xy}{\sqrt{12}v_1 v_2}$	1
(d)	$\frac{xy}{v_1 v_2}$	1	$p_x^{(1)} p_y^{(1)}$	Eq. (75)	$\frac{xy}{4v_1 v_2}$	$\frac{0.22xy}{v_1 v_2}$	1
		2	$p_x^{(1)} p_y^{(2)}$	Eq. (75)	$\frac{xy}{4v_1 v_2}$	$\frac{0.22xy}{v_1 v_2}$	1
		3	$p_x^{(2)} p_y^{(1)}$	Eq. (75)	$\frac{xy}{4v_1 v_2}$	$\frac{0.22xy}{v_1 v_2}$	1
		4	$p_x^{(2)} p_y^{(2)}$	Eq. (75)	$\frac{xy}{4v_1 v_2}$	$\frac{0.22xy}{v_1 v_2}$	1

Table 7: The fraction of the voxel covered for each of the partially covered voxels, its distribution, expected value and standard deviation, along with the number of voxels with that fraction covered, for the case of 2 small dimensions in the 2-dimensional setting (i.e., $x < v_1$ and $y < v_2$).

Subcase S	Region R	$g(F \mid R, S)$	$p(R \mid S)$
(a)	1	constant at $\frac{xy}{v_1 v_2}$	1
(b)	1	$g(p_y^{(1)} \frac{x}{v_1})$	1/2
	2	$g(p_y^{(2)} \frac{x}{v_1})$	1/2
(c)	1	$g(p_x^{(1)} \frac{y}{v_2})$	1/2
	2	$g(p_x^{(2)} \frac{y}{v_2})$	1/2
(d)	1	$g(p_x^{(1)} p_y^{(1)})$	1/4
	2	$g(p_x^{(1)} p_y^{(2)})$	1/4
	3	$g(p_x^{(2)} p_y^{(1)})$	1/4
	4	$g(p_x^{(2)} p_y^{(2)})$	1/4

Table 8: The general expressions for $g(F \mid R, S)$ and $p(R \mid S)$ in the right-hand side of (77).

1) Generate the subcase S (a), (b), (c), or (d) according to their probabilities in Table 7 (using Uniform (0,1) random variables, as discussed in Appendix C.2).

2) If S = (a) in step 1,
   – set F equal to $(x/v_1) \cdot (y/v_2)$.

   If S = (b) in step 1,
   – generate $p_y^{(1)} \sim U(0, y/v_2)$,
   – set $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and
   – set F to be the sequence of the following 2 values:
   $$x/v_1 \cdot p_y^{(1)}, \quad x/v_1 \cdot p_y^{(2)}.$$

   If S = (c) in step 1,
   – generate $p_x^{(1)} \sim U(0, x/v_1)$,
   – set $p_x^{(2)} = x/v_1 - p_x^{(1)}$, and
   – set F to be the sequence of the following 2 values:
   $$y/v_2 \cdot p_x^{(1)}, \quad y/v_2 \cdot p_x^{(2)}.$$

   If S = (d) in step 1,
   – generate $p_x^{(1)} \sim U(0, x/v_1)$ and $p_y^{(1)} \sim U(0, y/v_2)$,
   – set $p_x^{(2)} = x/v_1 - p_x^{(1)}$, and $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and
   – set F to be the sequence of the following 4 values:
   $$p_x^{(1)} \cdot p_y^{(1)}, \quad p_x^{(1)} \cdot p_y^{(2)}, \quad p_x^{(2)} \cdot p_y^{(1)}, \quad p_x^{(2)} \cdot p_y^{(2)}.$$

3) Repeat steps 1–2 K times for a large value K (e.g., 100000).

   The resulting set of K sequences F will be a sample from the density in (77). This set can then be queried for any summary quantity of interest, such as quantiles, percentiles, means, medians and variances of the fractions covered in all the partially covered voxels.

Table 9: Algorithm for generating values of the fraction covered, F, from the distribution in (77) in the 2-dimensional setting with 2 small dimensions ($x < v_1$ and $y < v_2$).

Variable	**Notation**	**Expression**	**Distribution**	**Expected Value**	**Standard Deviation**
# Fully Covered Voxels	N_f	Eq. (43)	Eq. (44)	Eq. (45)	Sq. root of (46)
# Partially Covered Voxels	N_p	Eq. (47)	Eq. (48)	Eq. (49)	Sq. root of (50)
Fractions Covered In Partially Covered Voxels	see Tables 1 and 3				

Table 10: Summary of the variables of interest in the 2-dimensional setting for the case when the sheet dimensions are both greater than the corresponding voxel dimensions (i.e., $\boldsymbol{x \geq v_1}$ **and** $\boldsymbol{y \geq v_2}$).

Variable	**Notation**	**Expression**	**Distribution**	**Expected Value**	**Standard Deviation**
# Fully Covered Voxels	N_f	0	fixed at 0	0	0
# Partially Covered Voxels	N_p	Eq. (61)	Eq. (62)	$\left(1+\frac{x}{v_1}\right)\left(1+\frac{y}{v_2}\right)$	Sq. root of (64)
Fractions Covered In Partially Covered Voxels	see Tables 4 and 6				

Table 11: Summary of the variables of interest in the 2-dimensional setting for the case of 1 small dimension. The small dimension is denoted with y, so when $x \geq v_1$ and $y < v_2$.

Variable	**Notation**	**Expression**	**Distribution**	**Expected Value**	**Standard Deviation**
# Fully Covered Voxels	N_f	0	fixed at 0	0	0
# Partially Covered Voxels	N_p	Eq. (65)	Eq. (66)	$\left(1+\frac{x}{v_1}\right)\left(1+\frac{y}{v_2}\right)$	Sq. root of (68)
Fractions Covered In Partially Covered Voxels	see Tables 7 and 9				

Table 12: Summary of the variables of interest in the 2-dimensional setting for the case of 2 small dimensions, i.e., $x < v_1$ and $y < v_2$.

3 Three-Dimensional Setting

All the results developed in for the 2-dimensional setting in Section 2 can now be easily extended to the 3-dimensional setting. Assume a rectangular object with edge length x along the first dimension, edge length y along the second dimension, and edge length z along the third dimension, randomly centered at location (c_x, c_y, c_z) in a voxel. Assume rectangular voxels, each with edge length v_1 along the first dimension, edge length v_2 along the second dimension and edge length v_3 along the third dimension.

The quantities x, y, z, v_1, v_2 and v_3 are assumed to be known. Analogously to the 2-dimensional setting, randomly placing the center of the object in a voxel means that the location (c_x, c_y, c_z) is a random variable with the Uniform distribution on a rectangular region $(0, v_1) \times (0, v_2) \times (0, v_3)$. This is equivalent to sampling $c_x \sim$ Uniform(0, v_1), $c_y \sim$ Uniform(0, v_2) and $c_z \sim$ Uniform(0,1) independently from each other. The object is assumed to fit completely in the field of view, so the total number of voxels covered by the object is given by $xyz/(v_1 v_2 v_3)$, with the non-integer part reflecting the fact that some voxels are only partially covered.

As in Section 2, we will first assume that $x \geq v_1$, $y \geq v_2$ and $z \geq v_3$. Section 3.5 covers the special cases when at least one of the object dimensions is smaller than the corresponding voxel dimension.

3.1 Notation and Remaining Preliminaries

The variables f_x and f_y are defined in (1) and (30), and their distributions are given in (19)–(20) and (31)–(32), respectively. An analogous random variable in the third dimension, f_z, is given by

$$f_z = \left\lfloor \frac{c_z + z/2}{v_3} \right\rfloor - \left\lceil \frac{c_z - z/2}{v_3} \right\rceil . \tag{80}$$

Letting k_z and m_z be defined as

$$k_z \equiv \left\lfloor \frac{z/2}{v_3} \right\rfloor \tag{81}$$

and

$$m_z \equiv frac\left(\frac{z/2}{v_3}\right) = \frac{z/2}{v_3} - k_z, \tag{82}$$

following analogous steps to those for deriving the distribution of f_x, the distribution of f_z is then obtained as follows:

$$\begin{aligned} &\text{if} \quad m_z < 0.5, \\ f_z &= \begin{cases} 2k_z & \text{with probability } 2m_z \\ 2k_z - 1 & \text{with probability } 1 - 2m_z \end{cases} \end{aligned} \tag{83}$$

while

$$\begin{aligned} \text{if} \quad & m_z \geq 0.5, \\ f_z \quad &= \begin{cases} 2k_z & \text{with probability } 2(1-m_z) \\ 2k_z+1 & \text{with probability } 2m_z-1 \end{cases} \end{aligned} \tag{84}$$

The distributions in (83) and (84) can be alternatively expressed as follows:

$$f_z = \begin{cases} f_z^{(1)} & \text{with probability } p(f_z^{(1)}) \\ f_z^{(2)} & \text{with probability } p(f_z^{(2)}) \end{cases} \tag{85}$$

where

$$\begin{aligned} f_z^{(1)} &= 2k_z, \\ f_z^{(2)} &= \begin{cases} 2k_z-1 & \text{if } m_z < 0.5 \\ 2k_z+1 & \text{if } m_z \geq 0.5 \end{cases} \end{aligned} \tag{86}$$

$$\begin{aligned} p(f_z^{(1)}) &= \begin{cases} 2m_z & \text{if } m_z < 0.5 \\ 2(1-m_z) & \text{if } m_z \geq 0.5 \end{cases} \\ p(f_z^{(2)}) &= 1 - p(f_z^{(1)}). \end{aligned} \tag{87}$$

Analogously to f_x and f_y, the expected value of f_z is given by

$$E(f_z) = \frac{z}{v_3} - 1, \tag{88}$$

while the variance of f_z is given by

$$Var(f_z) = 4 \cdot |m_z - 0.5| \cdot (0.5 - |m_z - 0.5|) \tag{89}$$

for $0 \leq m_z < 1$. The standard deviation of f_z is the square root of the variance of f_z, i.e., it is given by

$$\sigma_{f_z} = 2\sqrt{|m_z - 0.5| \cdot (0.5 - |m_z - 0.5|)}. \tag{90}$$

Also, define

$$p_z^{(1)} = \left\lceil \frac{c_z - z/2}{v_3} \right\rceil - \frac{c_z - z/2}{v_3} \tag{91}$$

and

$$p_z^{(2)} = \frac{c_z + z/2}{v_3} - \left\lfloor \frac{c_z + z/2}{v_3} \right\rfloor. \tag{92}$$

Since these are exactly equivalent to their counterparts $p_x^{(1)}$ and $p_x^{(2)}$ (as well as $p_y^{(1)}$ and $p_y^{(2)}$), they each follow the Uniform(0,1) distribution.

3.2 Number of Fully Covered Voxels N_f

Analogously to the 2-dimensional setting, the number of fully covered voxels in the 3-dimensional setting, still denoted by N_f, is just the product of the number of fully covered voxels in each dimension. That is, $N_f = f_x f_y f_z$ with f_x, f_y, and f_z defined in (1), (30) and (80), or

$$\begin{aligned} N_f &= f_x f_y f_z \\ &= \left(\left\lfloor \frac{c_x + x/2}{v_1} \right\rfloor - \left\lceil \frac{c_x - x/2}{v_1} \right\rceil \right) \\ &\times \left(\left\lfloor \frac{c_y + y/2}{v_2} \right\rfloor - \left\lceil \frac{c_y - y/2}{v_2} \right\rceil \right) \\ &\times \left(\left\lfloor \frac{c_z + z/2}{v_3} \right\rfloor - \left\lceil \frac{c_z - z/2}{v_3} \right\rceil \right) \end{aligned} \tag{93}$$

Since each of f_x, f_y, and f_z takes on 2 possible values for a given set of x, y, z, v_1, v_2 and v_3 values, N_f takes on 8 possible values.

Since f_x, f_y, and f_z are independent of each other due to the independence of c_x, c_y, and c_z, the probabilities associated with each of the 8 possible values are just the products of the probabilities of each possible value in the 1-dimensional setting. Thus, we have the following 8-point distribution for N_f:

$$N_f = \begin{cases} f_x^{(1)} \cdot f_y^{(1)} \cdot f_z^{(1)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(1)}) \\ f_x^{(1)} \cdot f_y^{(1)} \cdot f_z^{(2)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(2)}) \\ f_x^{(1)} \cdot f_y^{(2)} \cdot f_z^{(1)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(1)}) \\ f_x^{(1)} \cdot f_y^{(2)} \cdot f_z^{(2)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(2)}) \\ f_x^{(2)} \cdot f_y^{(1)} \cdot f_z^{(1)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(1)}) \\ f_x^{(2)} \cdot f_y^{(1)} \cdot f_z^{(2)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(2)}) \\ f_x^{(2)} \cdot f_y^{(2)} \cdot f_z^{(1)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(1)}) \\ f_x^{(2)} \cdot f_y^{(2)} \cdot f_z^{(2)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(2)}) \end{cases} \tag{94}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22), $f_y^{(1)}$ and $f_y^{(2)}$ as defined in (36), $f_z^{(1)}$ and $f_z^{(2)}$ as defined in (86), $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23), $p(f_y^{(1)})$ and $p(f_y^{(2)})$ as defined in (37), and $p(f_z^{(1)})$ and $p(f_z^{(2)})$ as defined in (87).

Since $N_f = f_x f_y f_z$ and f_x, f_y and f_z are independent, the expected value of N_f in the 3-dimensional setting is given by

$$E(N_f) = E(f_x f_y f_z) = E(f_x)E(f_y)E(f_z) = \left(\frac{x}{v_1} - 1 \right) \left(\frac{y}{v_2} - 1 \right) \left(\frac{z}{v_3} - 1 \right). \tag{95}$$

The independence of f_x, f_y and f_z also implies the independence of f_x^2, f_y^2 and f_z^2. This, put

together with (24), (25), (38), (39), (88), and (89), gives that the variance of N_f is equal to

$$\begin{aligned}
Var(N_f) &= E(N_f^2) - (E(N_f))^2 = E(f_x^2 f_y^2 f_z^2) - (E(f_x f_y f_z))^2 \\
&= E(f_x^2)E(f_y^2) - (E(f_x)E(f_y)E(f_z))^2 \\
&= \left[Var(f_x) + (E(f_x))^2\right] \cdot \left[Var(f_y) + (E(f_y))^2\right] \cdot \left[Var(f_z) + (E(f_z))^2\right] \\
&- (E(f_x))^2 (E(f_y))^2 (E(f_z))^2 \\
&= \left[4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) + \left(\frac{x}{v_1} - 1\right)^2\right] \\
&\times \left[4 \cdot |m_y - 0.5| \cdot (0.5 - |m_y - 0.5|) + \left(\frac{y}{v_2} - 1\right)^2\right] \\
&\times \left[4 \cdot |m_z - 0.5| \cdot (0.5 - |m_z - 0.5|) + \left(\frac{z}{v_3} - 1\right)^2\right] \\
&- \left(\frac{x}{v_1} - 1\right)^2 \left(\frac{y}{v_2} - 1\right)^2 \left(\frac{z}{v_3} - 1\right)^2
\end{aligned} \tag{96}$$

The standard deviation of the number of fully covered voxels is the square root of the expression in (96).

3.3 Number of Partially Covered Voxels N_p

In the 2-dimensional setting, the following contributed to the total number of partially covered voxels:

1. 1 voxel in each of the 4 corners of the object,
2. f_x voxels along each of the 2 horizontal edges (top and bottom) excluding the corners of item 1,
3. f_y voxels along each of the 2 vertical edges (left and right) excluding the corners of item 1.

In the 3-dimensional setting, the partially covered voxels are found in equivalent locations, plus in some new ones that did not exist in the 2-dimensional setting. Specifically, there are 26 regions of partially covered voxels, comprising of the following:

1. Regions 1–8: 1 voxel in each of the 8 corners of the object,
2. Regions 9–12: f_x voxels along each of the 4 horizontal edges (top and bottom, front and back) excluding the voxels of item 1,
3. Regions 13–16: f_y voxels along each of the 4 vertical edges (left and right, front and back) excluding the voxels of item 1,

4. Regions 17–20: f_z voxels along each of the 4 front-back edges (left and right, top and bottom) excluding the voxels of item 1,

5. Regions 21, 22: $f_x f_y$ voxels on each of the front and back faces excluding the voxels of items 1, 2, 3,

6. Regions 23, 24: $f_x f_z$ voxels on each of the top and bottom faces excluding the voxels of items 1, 2, 4,

7. Regions 25, 26: $f_y f_z$ voxels on each of the left and right faces excluding the voxels of items 1, 3, 4,

(Note that regions 1–4 above are the 3-dimensional equivalents of regions 1–4 in Figure 3, regions 9 and 10 above are the 3-dimensional equivalents of regions 5 and 6 in Figure 3, while regions 13 and 14 above are the 3-dimensional equivalents of regions 7 and 8 in Figure 3.)

Thus, the number of partially covered voxels is equal to

$$N_p = 8 + 4(f_x + f_y + f_z) + 2(f_x f_y + f_x f_z + f_y f_z). \tag{97}$$

Since N_p is only a function of f_x, f_y and f_z, each of which can take on 2 possible values, N_p takes on 8 possible values. Moreover, since f_x, f_y and f_z are independent, the probability of any given value resulting from a specific combination of these three variables is just the product of their individual probabilities in the one-dimensional setting. Therefore, the distribution of N_p is given as follows:

$$N_p = \begin{cases} 8 + 4 \cdot (f_x^{(1)} + f_y^{(1)} + f_z^{(1)}) + 2 \cdot (f_x^{(1)} f_y^{(1)} + f_x^{(1)} f_z^{(1)} + f_y^{(1)} f_z^{(1)}) & \text{with pr. } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(1)}) \\ 8 + 4 \cdot (f_x^{(1)} + f_y^{(1)} + f_z^{(2)}) + 2 \cdot (f_x^{(1)} f_y^{(1)} + f_x^{(1)} f_z^{(2)} + f_y^{(1)} f_z^{(2)}) & \text{with pr. } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(2)}) \\ 8 + 4 \cdot (f_x^{(1)} + f_y^{(2)} + f_z^{(1)}) + 2 \cdot (f_x^{(1)} f_y^{(2)} + f_x^{(1)} f_z^{(1)} + f_y^{(2)} f_z^{(1)}) & \text{with pr. } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(1)}) \\ 8 + 4 \cdot (f_x^{(1)} + f_y^{(2)} + f_z^{(2)}) + 2 \cdot (f_x^{(1)} f_y^{(2)} + f_x^{(1)} f_z^{(2)} + f_y^{(2)} f_z^{(2)}) & \text{with pr. } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(2)}) \\ 8 + 4 \cdot (f_x^{(2)} + f_y^{(1)} + f_z^{(1)}) + 2 \cdot (f_x^{(2)} f_y^{(1)} + f_x^{(2)} f_z^{(1)} + f_y^{(1)} f_z^{(1)}) & \text{with pr. } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(1)}) \\ 8 + 4 \cdot (f_x^{(2)} + f_y^{(1)} + f_z^{(2)}) + 2 \cdot (f_x^{(2)} f_y^{(1)} + f_x^{(2)} f_z^{(2)} + f_y^{(1)} f_z^{(2)}) & \text{with pr. } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot p(f_z^{(2)}) \\ 8 + 4 \cdot (f_x^{(2)} + f_y^{(2)} + f_z^{(1)}) + 2 \cdot (f_x^{(2)} f_y^{(2)} + f_x^{(2)} f_z^{(1)} + f_y^{(2)} f_z^{(1)}) & \text{with pr. } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(1)}) \\ 8 + 4 \cdot (f_x^{(2)} + f_y^{(2)} + f_z^{(2)}) + 2 \cdot (f_x^{(2)} f_y^{(2)} + f_x^{(2)} f_z^{(2)} + f_y^{(2)} f_z^{(2)}) & \text{with pr. } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot p(f_z^{(2)}) \end{cases} \tag{98}$$

Using the independence of f_x, f_y, and f_z again, we have that the expected value of N_p is equal to

$$\begin{aligned} E(N_p) &= 8 + 4 \cdot [E(f_x) + E(f_y) + E(f_z)] + 2 \cdot [E(f_x)E(f_y) + E(f_x)E(f_z) + E(f_y)E(f_z)] \\ &= 8 + 4\left(\frac{x}{v_1} - 1 + \frac{y}{v_2} - 1 + \frac{z}{v_3} - 1\right) \\ &+ 2\left[\left(\frac{x}{v_1} - 1\right)\left(\frac{y}{v_2} - 1\right) + \left(\frac{x}{v_1} - 1\right)\left(\frac{z}{v_3} - 1\right) + \left(\frac{y}{v_2} - 1\right)\left(\frac{z}{v_3} - 1\right)\right] \\ &= 2\left(\frac{x}{v_1}\frac{y}{v_2} + \frac{x}{v_1}\frac{z}{v_3} + \frac{y}{v_2}\frac{z}{v_3} + 1\right). \end{aligned} \tag{99}$$

As shown in Appendix G, the variance of N_p is equal to

$$\begin{aligned} Var(N_p) &= 4\left(Var(f_x f_y) + Var(f_x f_z) + Var(f_y f_z)\right) \\ &+ 8\left[Var(f_x)\left(\frac{y}{v_2}\frac{z}{v_3} + \frac{y}{v_2} + \frac{z}{v_3} - 1\right) + Var(f_y)\left(\frac{x}{v_1}\frac{z}{v_3} + \frac{x}{v_1} + \frac{z}{v_3} - 1\right)\right. \\ &+ \left. Var(f_z)\left(\frac{x}{v_1}\frac{y}{v_2} + \frac{x}{v_1} + \frac{y}{v_2} - 1\right)\right]. \end{aligned} \tag{100}$$

The standard deviation of N_p is just the square root of the expression in (100).

3.4 Fractions Covered in Partially Covered Voxels

As described in Section 3.3, there are 26 distinct regions of partially covered voxels in a 3-dimensional rectangular object. Table 13 lists each of these regions, its location in the object, each region's value of the fraction covered, its distribution, expected value and standard deviation, as well as the number of partially covered voxels in that region, its distribution, expected value and standard deviation.

(As a check, note that the sum of the fractions covered in regions 1–8 is equal to

$$\left(p_x^{(1)} + p_x^{(2)}\right)\left(p_y^{(1)} + p_y^{(2)}\right)\left(p_z^{(1)} + p_z^{(2)}\right), \tag{101}$$

the sum of the fractions covered in regions 9–12 is equal to

$$f_x\left(p_y^{(1)} + p_y^{(2)}\right)\left(p_z^{(1)} + p_z^{(2)}\right), \tag{102}$$

the sum of the fractions covered in regions 13–16 is equal to

$$f_y\left(p_x^{(1)} + p_x^{(2)}\right)\left(p_z^{(1)} + p_z^{(2)}\right), \tag{103}$$

the sum of the fractions covered in regions 17–20 is equal to

$$f_z\left(p_x^{(1)} + p_x^{(2)}\right)\left(p_y^{(1)} + p_y^{(2)}\right), \tag{104}$$

while the sum of the fractions covered in regions 21–26 is equal to

$$f_x f_y\left(p_z^{(1)} + p_z^{(2)}\right) + f_x f_z\left(p_y^{(1)} + p_y^{(2)}\right) + f_y f_z\left(p_x^{(1)} + p_x^{(2)}\right). \tag{105}$$

Adding (101)–(105) and the number of fully covered voxels $N_f = f_x f_y f_z$ gives the sum

$$\left(p_x^{(1)} + p_x^{(2)} + f_x\right)\left(p_y^{(1)} + p_y^{(2)} + f_y\right)\left(p_z^{(1)} + p_z^{(2)} + f_z\right) = \left(\frac{x}{v_1}\right)\left(\frac{y}{v_2}\right)\left(\frac{z}{v_3}\right) = \frac{xyz}{v_1 v_2 v_3}, \tag{106}$$

the total number of voxels covered by the object, as required.)

Now, as for the distributions of the fractions covered in the partially covered voxels in the 3rd column of Table 13, we have already derived the distributions of all but those in regions 1–8. The distributions of fractions in regions 21–26, $p_x^{(1)}$, $p_x^{(2)}$, $p_y^{(1)}$, $p_y^{(2)}$, $p_z^{(1)}$, $p_z^{(2)}$,

Region	Region Description	Fraction covered				# Voxels at That Fraction Covered			
		Expression	Distribution	Exp. Value	Std. Dev.	Expression	Distribution	Exp. Value	Std. Dev.
1	Lt btm corner of frt face	$p_x^{(1)} p_y^{(1)} p_z^{(1)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
2	Rt btm corner of frt face	$p_x^{(2)} p_y^{(1)} p_z^{(1)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
3	Lt top corner of frt face	$p_x^{(1)} p_y^{(2)} p_z^{(1)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
4	Rt top corner of frt face	$p_x^{(2)} p_y^{(2)} p_z^{(1)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
5	Lt btm corner of back face	$p_x^{(1)} p_y^{(1)} p_z^{(2)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
6	Rt btm corner of back face	$p_x^{(2)} p_y^{(1)} p_z^{(2)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
7	Lt top corner of back face	$p_x^{(1)} p_y^{(2)} p_z^{(2)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
8	Rt top corner of back face	$p_x^{(2)} p_y^{(2)} p_z^{(2)}$	Eq. (108)	1/8	0.15	1	n/a	1	0
9	Btm horiz. edge of frt face excl. regions 1, 2	$p_y^{(1)} p_z^{(1)}$	Eq. (52)	1/4	0.22	f_x	Eq. (19)-(20)	$x/v_1 - 1$	Eq. (26)
10	Top horiz. edge of frt face excl. regions 3, 4	$p_y^{(2)} p_z^{(1)}$	Eq. (52)	1/4	0.22	f_x	Eq. (19)-(20)	$x/v_1 - 1$	Eq. (26)
11	Btm horiz. edge of back face excl. regions 5, 6	$p_y^{(1)} p_z^{(2)}$	Eq. (52)	1/4	0.22	f_x	Eq. (19)-(20)	$x/v_1 - 1$	Eq. (26)
12	Top horiz. edge of back face excl. regions 7, 8	$p_y^{(2)} p_z^{(2)}$	Eq. (52)	1/4	0.22	f_x	Eq. (19)-(20)	$x/v_1 - 1$	Eq. (26)
13	Lt vert. edge of frt face excl. regions 1, 3	$p_x^{(1)} p_z^{(1)}$	Eq. (52)	1/4	0.22	f_y	Eq. (31)-(32)	$y/v_2 - 1$	Eq. (40)
14	Rt vert. edge of frt face excl. regions 2, 4	$p_x^{(2)} p_z^{(1)}$	Eq. (52)	1/4	0.22	f_y	Eq. (31)-(32)	$y/v_2 - 1$	Eq. (40)
15	Lt vert. edge of back face excl. regions 5, 7	$p_x^{(1)} p_z^{(2)}$	Eq. (52)	1/4	0.22	f_y	Eq. (31)-(32)	$y/v_2 - 1$	Eq. (40)
16	Rt vert. edge of back face excl. regions 6, 8	$p_x^{(2)} p_z^{(2)}$	Eq. (52)	1/4	0.22	f_y	Eq. (31)-(32)	$y/v_2 - 1$	Eq. (40)
17	Btm frt-back edge of lt face excl. regions 1, 5	$p_x^{(1)} p_y^{(1)}$	Eq. (52)	1/4	0.22	f_z	Eq. (83)-(84)	$z/v_3 - 1$	Eq. (90)
18	Btm frt-back edge of rt face excl. regions 2, 6	$p_x^{(2)} p_y^{(1)}$	Eq. (52)	1/4	0.22	f_z	Eq. (83)-(84)	$z/v_3 - 1$	Eq. (90)
19	Top frt-back edge of lt face excl. regions 3, 7	$p_x^{(1)} p_y^{(2)}$	Eq. (52)	1/4	0.22	f_z	Eq. (83)-(84)	$z/v_3 - 1$	Eq. (90)
20	Top frt-back edge of rt face excl. regions 4, 8	$p_x^{(2)} p_y^{(2)}$	Eq. (52)	1/4	0.22	f_z	Eq. (83)-(84)	$z/v_3 - 1$	Eq. (90)
21	Frt face excl. regions 1–4, 9, 10, 13, 14	$p_z^{(1)}$	U(0,1)	1/2	0.29	$f_x f_y$	Eq. (110)	Eq. (113)	Sq. root of (114)
22	Back face excl. regions 5–8, 11, 12, 15, 16	$p_z^{(2)}$	U(0,1)	1/2	0.29	$f_x f_y$	Eq. (110)	Eq. (113)	Sq. root of (114)
23	Btm face excl. regions 1, 2, 5, 6, 9, 11, 17, 18	$p_y^{(1)}$	U(0,1)	1/2	0.29	$f_x f_z$	Eq. (111)	Eq. (115)	Sq. root of (116)
24	Top face excl. regions 3, 4, 7, 8, 10, 12, 19, 20	$p_y^{(2)}$	U(0,1)	1/2	0.29	$f_x f_z$	Eq. (111)	Eq. (115)	Sq. root of (116)
25	Lt face excl. regions 1, 3, 5, 7, 13, 15, 17, 19	$p_x^{(1)}$	U(0,1)	1/2	0.29	$f_y f_z$	Eq. (112)	Eq. (117)	Sq. root of (118)
26	Rt face excl. regions 2, 4, 6, 8, 14, 16, 18, 20	$p_x^{(2)}$	U(0,1)	1/2	0.29	$f_y f_z$	Eq. (112)	Eq. (117	Sq. root of (118)

Table 13: The location of each of the 26 regions of partially covered voxels in a 3-dimensional rectangular object (lt = left, rt = right, btm = bottom, frt = front, excl. = excluding, horiz. = horizontal, vert. = vertical), the fraction of the voxel covered in each region of the partially covered voxels, its distribution, expected value and standard deviation, and the number of voxels with that fraction covered, its distribution, expected value and standard deviation. Note that regions 1–4 above are the 3-dimensional equivalents of regions 1–4 in Figure 3, regions 9 and 10 above are the 3-dimensional equivalents of regions 5 and 6 in Figure 3, and regions 13 and 14 above are the 3-dimensional equivalents of regions 7 and 8 in Figure 3.

have been shown to be Uniform(0,1) in Sections 2.1 and 3.1. The expected (mean) value, the median and the standard deviation of each of these 6 fractions are therefore 1/2, 1/2, and $1/\sqrt{12} = 0.2887$, respectively. The probability that any of the six of these fractions is between values a and b (with $0 \leq a < b \leq 1$) is simply the length of that interval, $b - a$. For example, the probability that the fraction of the voxel covered in Region 26 is between 0.2 and 0.5 is equal to 0.3.

The distributions of the fractions covered in regions 9–20 in Table 13 have also been previously discussed in Section 2.2.3 and have each been shown (Appendix F.2) to have the the probability density function given by

$$g_W(w) = -log\ w, \qquad 0 \leq w \leq 1 \tag{107}$$

and 0 otherwise, where w is the particular value of each of the 12 fractions.

The fractions covered in regions 1–8 in Table 13 are products of three Uniform(0,1) random variables. As shown in Section F.3, each of these has the following probability density function (pdf), denoted by h_W:

$$h_W(w) = \frac{(log\ w)^2}{2}, \qquad 0 \leq w \leq 1 \tag{108}$$

and 0 otherwise, where w denotes the particular value of each of the 8 fractions. Thus, to obtain the probability that any of the 8 of these fractions is in a certain interval, one needs to integrate the function in (108) over that interval. Specifically, as shown in Section F.3, the probability that any of these four fractions is between values a and b (with $0 \leq a < b \leq 1$) is equal to

$$P(a \leq W \leq b) = \frac{b\left((log\ b - 1)^2 + 1\right) - a\left((log\ a - 1)^2 + 1\right)}{2} \tag{109}$$

with $0 \cdot (log\ 0 - 1)^2 \equiv 0$. As shown in Section F.3, the expectation, the median, and the standard deviation of each of the 8 fractions in regions 1–8 of Table 13 are 0.125, 0.069, and 0.1463, respectively.

As for the number of voxels at each fraction covered, as shown in Table 13, it is fixed at 1 in regions 1–8 and is equal to f_x in regions 9–12, f_y in regions 13–16, f_z in regions 17–20, $f_x f_y$ in regions 21 and 22, $f_x f_z$ in regions 23 and 24 and $f_y f_z$ in regions 25 and 26. The distributions, the means and the standard deviations of f_x, f_y and f_z have been previously derived and are shown in Table 13. The distribution, the mean and standard deviation of $f_x f_y$ was derived, as well, as it is exactly the expression for the number of fully covered voxels N_f in the 2-dimensional setting. Specifically, as shown in (44), the distribution of $f_x f_y$ is given as follows:

$$f_x f_y = \begin{cases} f_x^{(1)} \cdot f_y^{(1)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \\ f_x^{(1)} \cdot f_y^{(2)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \\ f_x^{(2)} \cdot f_y^{(1)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \\ f_x^{(2)} \cdot f_y^{(2)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \end{cases} \tag{110}$$

Similarly, the distributions of $f_x f_z$ and $f_y f_z$ are given by

$$f_x f_z = \begin{cases} f_x^{(1)} \cdot f_z^{(1)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_z^{(1)}) \\ f_x^{(1)} \cdot f_z^{(2)} & \text{with probability } p(f_x^{(1)}) \cdot p(f_z^{(2)}) \\ f_x^{(2)} \cdot f_z^{(1)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_z^{(1)}) \\ f_x^{(2)} \cdot f_z^{(2)} & \text{with probability } p(f_x^{(2)}) \cdot p(f_z^{(2)}) \end{cases} \tag{111}$$

$$f_y f_z = \begin{cases} f_y^{(1)} \cdot f_z^{(1)} & \text{with probability } p(f_y^{(1)}) \cdot p(f_z^{(1)}) \\ f_y^{(1)} \cdot f_z^{(2)} & \text{with probability } p(f_y^{(1)}) \cdot p(f_z^{(2)}) \\ f_y^{(2)} \cdot f_z^{(1)} & \text{with probability } p(f_y^{(2)}) \cdot p(f_z^{(1)}) \\ f_y^{(2)} \cdot f_z^{(2)} & \text{with probability } p(f_y^{(2)}) \cdot p(f_z^{(2)}) \end{cases} \tag{112}$$

In (45), the mean of $f_x f_y$ was shown to be equal to

$$E(f_x f_y) = \left(\frac{x}{v_1} - 1\right) \cdot \left(\frac{y}{v_2} - 1\right), \tag{113}$$

while the variance of $f_x f_y$ was shown in (46) to be equal to

$$\begin{aligned} Var(f_x f_y) &= \left[4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) + \left(\frac{x}{v_1} - 1\right)^2\right] \\ &\times \left[4 \cdot |m_y - 0.5| \cdot (0.5 - |m_y - 0.5|) + \left(\frac{y}{v_2} - 1\right)^2\right] \\ &- \left(\frac{x}{v_1} - 1\right)^2 \left(\frac{y}{v_2} - 1\right)^2 \end{aligned} \tag{114}$$

Analogously, the mean of $f_x f_z$ is given by

$$E(f_x f_z) = \left(\frac{x}{v_1} - 1\right) \cdot \left(\frac{z}{v_3} - 1\right), \tag{115}$$

while the variance of $f_x f_z$ is equal to

$$\begin{aligned} Var(f_x f_z) &= \left[4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) + \left(\frac{x}{v_1} - 1\right)^2\right] \\ &\times \left[4 \cdot |m_z - 0.5| \cdot (0.5 - |m_z - 0.5|) + \left(\frac{z}{v_3} - 1\right)^2\right] \\ &- \left(\frac{x}{v_1} - 1\right)^2 \left(\frac{z}{v_3} - 1\right)^2 \end{aligned} \tag{116}$$

The mean of $f_y f_z$ is given by

$$E(f_y f_z) = \left(\frac{y}{v_2} - 1\right) \cdot \left(\frac{z}{v_3} - 1\right), \tag{117}$$

while the variance of $f_y f_z$ is equal to

$$\begin{aligned} Var(f_y f_z) &= \left[4 \cdot |m_y - 0.5| \cdot (0.5 - |m_y - 0.5|) + \left(\frac{y}{v_2} - 1\right)^2\right] \\ &\times \left[4 \cdot |m_z - 0.5| \cdot (0.5 - |m_z - 0.5|) + \left(\frac{z}{v_3} - 1\right)^2\right] \\ &- \left(\frac{y}{v_2} - 1\right)^2 \left(\frac{z}{v_3} - 1\right)^2 \end{aligned} \tag{118}$$

As discussed in Section 2.2.3, one may be more interested in the *entire* distribution of fractions covered in all partially covered voxels without fixing the region. That is, the question of interest may be about the fraction covered by a randomly picked partially covered voxel whatever region it may be in. This distribution is a *random mixture* of the distributions within each of the 26 regions in Table 13, with the mixing fractions equal to the relative frequencies of each of the 26 regions. Note that the relative frequency of each region is the number in the 7th column of Table 13 (i.e., 1 for regions 1–8, f_x for regions 9–12, f_y for regions 13–16, f_z for regions 17–20, $f_x f_y$ for regions 21 and 22, $f_x f_z$ for regions 23 and 24, and $f_y f_z$ for regions 25 and 26) divided by the total number of partially covered voxels $N_p = 8 + 4(f_x + f_y + f_z) + 2(f_x f_y + f_x f_z + f_y f_z)$. Thus, the mixing fractions are themselves random, as they are functions of the random variables f_x, f_y and f_z. Extending the equation in (55) to the 3-dimensional setting, we have that the density for the fraction covered F is given by

$$g(F) = \sum_R \sum_{f_x} \sum_{f_y} \sum_{f_z} g(F \mid R, f_x, f_y, f_z) \cdot p(R \mid f_x, f_y, f_z) \cdot p(f_x) \cdot p(f_y) \cdot p(f_z) \tag{119}$$

where

R denotes the region (1 through 26),

$g(F \mid R, f_x, f_y, f_z)$ denotes the conditional density of the fraction covered in a randomly picked voxel in a given region R and given values of f_x, f_y and f_z,

$p(R \mid f_x, f_y, f_z)$ is the relative frequency of the region R for given values of f_x, f_y and f_z, and

$p(f_x)$ is given in (19)–(20), $p(f_y)$ is given in (31)–(32), and $p(f_z)$ is given in (83)–(84),

Extending the procedure outlined in Table 3 to the 3-dimensional setting and following Table 13, the procedure for generating from the density in (119) is given in Table 14.

1) Independently generate f_x according to (19)–(20), f_y according to (31)–(32) and f_z according to (83)–(84) (using Uniform(0,1) random variables, as discussed in Appendix C.2).

2) Given f_x, f_y and f_z generated in step 1, independently generate a value from the distribution of $p_x^{(1)}$ given f_x, a value from the distribution of $p_y^{(1)}$ given f_y, and a value from the distribution of $p_z^{(1)}$ given f_z according to (56)–(59).

3) Set $p_x^{(2)} = x/v_1 - f_x - p_x^{(1)}$, $p_y^{(2)} = y/v_2 - f_y - p_y^{(1)}$, and $p_z^{(2)} = z/v_3 - f_z - p_z^{(1)}$.

4) Set F to be a sequence of the following $8 + 4(f_x + f_y) + 2(f_x f_y + f_x f_z + f_y f_z)$ values:

$$p_x^{(1)}p_y^{(1)}p_z^{(1)},\ p_x^{(2)}p_y^{(1)}p_z^{(1)},\ p_x^{(1)}p_y^{(2)}p_z^{(1)},\ p_x^{(2)}p_y^{(2)}p_z^{(1)},$$
$$p_x^{(1)}p_y^{(1)}p_z^{(2)},\ p_x^{(2)}p_y^{(1)}p_z^{(2)},\ p_x^{(1)}p_y^{(2)}p_z^{(2)},\ p_x^{(2)}p_y^{(2)}p_z^{(2)},$$

$$\underbrace{p_y^{(1)}p_z^{(1)},\ p_y^{(2)}p_z^{(1)},\ p_y^{(1)}p_z^{(2)},\ p_y^{(2)}p_z^{(2)},\ldots,}_{\text{repeat } f_x \text{ times}}$$
$$\underbrace{p_x^{(1)}p_z^{(1)},\ p_x^{(2)}p_z^{(1)},\ p_x^{(1)}p_z^{(2)},\ p_x^{(2)}p_z^{(2)},\ldots,}_{\text{repeat } f_y \text{ times}}$$
$$\underbrace{p_x^{(1)}p_y^{(1)},\ p_x^{(2)}p_y^{(1)},\ p_x^{(1)}p_y^{(2)},\ p_x^{(2)}p_y^{(2)},\ldots,}_{\text{repeat } f_z \text{ times}}$$
$$\underbrace{p_z^{(1)},\ p_z^{(2)},\ldots}_{\text{repeat } f_x f_y \text{ times}},\quad \underbrace{p_y^{(1)},\ p_y^{(2)},\ldots}_{\text{repeat } f_x f_z \text{ times}},\quad \underbrace{p_x^{(1)},\ p_x^{(2)},\ldots}_{\text{repeat } f_y f_z \text{ times}}.$$

5) Repeat steps 1–4 K times for a large value K (e.g., 100000).

The resulting set of K sequences F will be a sample from the density in (119). This set can then be queried for any summary quantity of interest, such as quantiles, percentiles, means, medians and variances of the fractions covered in all the partially covered voxels.

Table 14: Algorithm for generating values of the fraction covered, F, from the distribution in (119) in the 3-dimensional setting when $x \geq v_1$, $y \geq v_2$ and $z \geq v_3$.

3.5 Object Dimensions Smaller Than Voxel Dimensions

In this section, we consider the situation when at least one of the object dimensions is smaller than the corresponding voxel dimension, i.e., at least one of the following is true: $x < v_1$, $y < v_2$, $z < v_3$. The results here are an extension of the results obtained in the 2-dimensional setting in Section 2.3.

As in the 2-dimensional setting, all the voxels covered by the object are partially covered. That is, the number of fully covered voxels N_f is identically 0. As for the number of partially covered voxels and their fractions covered, as discussed in Section 2.1, there are two subcases to consider in each dimension: the object could partially cover 1 or 2 voxel lengths in each dimension. As in Section 2.3, we will use the term "small dimension" to refer to an object dimension that is smaller than its corresponding voxel dimension. There are 3 groups of cases to consider, each with its own set of subcases:

1. 1 small dimension, i.e., one of the following is true:

   $x \geq v_1$, $y \geq v_2$, $z < v_3$

   $x \geq v_1$, $y < v_2$, $z \geq v_3$

   $x < v_1$, $y \geq v_2$, $z \geq v_3$

   (a) 1 voxel length is partially covered along the small dimension

   (b) 2 voxel lengths are partially covered along the small dimension

2. 2 small dimensions, i.e., one of the following is true:

   $x \geq v_1$, $y < v_2$, $z < v_3$

   $x < v_1$, $y \geq v_2$, $z < v_3$

   $x < v_1$, $y < v_2$, $z \geq v_3$

   (a) 1 voxel length is partially covered along the 1st small dimension and 1 voxel length is partially covered along the 2nd small dimension

   (b) 1 voxel length is partially covered along the 1st small dimension and 2 voxel lengths are partially covered along the 2nd small dimension

   (c) 2 voxel lengths are partially covered along the 1st small dimension and 1 voxel length is partially covered along the 2nd small dimension

   (d) 2 voxel lengths are partially covered along the 1st small dimension and 2 voxel lengths are partially covered along the 2nd small dimension

3. 3 small dimensions, i.e., $x < v_1$, $y < v_2$, $z < v_3$

   (a) 1 voxel length is partially covered along the x dimension, 1 voxel length is partially covered along the y dimension, 1 voxel length is partially covered along the z dimension

(b) 1 voxel length is partially covered along the x dimension, 1 voxel length is partially covered along the y dimension, 2 voxel lengths are partially covered along the z dimension

(c) 1 voxel length is partially covered along the x dimension, 2 voxel lengths are partially covered along the y dimension, 1 voxel length is partially covered along the z dimension

(d) 2 voxel lengths are partially covered along the x dimension, 1 voxel length is partially covered along the y dimension, 1 voxel length is partially covered along the z dimension

(e) 1 voxel length is partially covered along the x dimension, 2 voxel lengths are partially covered along the y dimension, 2 voxel lengths are partially covered along the z dimension

(f) 2 voxel lengths are partially covered along the x dimension, 1 voxel length is partially covered along the y dimension, 2 voxel lengths are partially covered along the z dimension

(g) 2 voxel lengths are partially covered along the x dimension, 2 voxel lengths are partially covered along the y dimension, 1 voxel length is partially covered along the z dimension

(h) 2 voxel lengths are partially covered along the x dimension, 2 voxel lengths are partially covered along the y dimension, 2 voxel lengths are partially covered along the z dimension

While there is no randomness associated with which of the 3 groups of cases (or which specific case within a group) will apply (since the cases are defined entirely by the voxel and object dimensions, which are assumed to be known), within each case, there is randomness associated with which of the subcases will occur (note that the probabilities of all the subcases within each case must add up to 1).

3.5.1 Number of Partially Filled Voxels N_p

1. 1 small dimension, i.e., one of the following is true:

   $x \geq v_1$, $y \geq v_2$, $z < v_3$

   $x \geq v_1$, $y < v_2$, $z \geq v_3$

   $x < v_1$, $y \geq v_2$, $z \geq v_3$

   Since all the dimensions are interchangeable, we will denote the small dimension with z and the other two dimensions with x and y. The voxel dimensions corresponding to x, y, and z are still denoted as v_1, v_2, and v_3, respectively. Thus, $x \geq v_1$, $y \geq v_2$, $z < v_3$. As shown in Appendix D.3,the number of partially covered voxels in this case is given by

$$N_p = \begin{cases} (f_x + 2)(f_y + 2) & \text{if } m_z \leq q_z \leq 1 - m_z \\ 2(f_x + 2)(f_y + 2) & \text{otherwise} \end{cases} \tag{120}$$

with $q_z = c_z/v_3$, as before. The distribution of N_p (also derived in Appendix D.3) is given by

$$N_p = \begin{cases} (f_x^{(1)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot (1 - z/v_3) \\ (f_x^{(1)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot (1 - z/v_3) \\ (f_x^{(2)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot (1 - z/v_3) \\ (f_x^{(2)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot (1 - z/v_3) \\ 2(f_x^{(1)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot (z/v_3) \\ 2(f_x^{(1)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot (z/v_3) \\ 2(f_x^{(2)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot (z/v_3) \\ 2(f_x^{(2)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot (z/v_3) \end{cases} \tag{121}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22), $f_y^{(1)}$ and $f_y^{(2)}$ as defined in (36), $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23) and $p(f_y^{(1)})$ and $p(f_y^{(2)})$ as defined in (37).

The expected value of N_p is given by

$$E(N_p) = \left(\frac{x}{v_1}+1\right)\left(\frac{y}{v_2}+1\right)\left(\frac{z}{v_3}+1\right). \tag{122}$$

The variance of N_p is given by

$$\begin{aligned} Var(N_p) &= \left(4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) + \left(\frac{x}{v_1}+1\right)^2\right) \\ &\times \left(4 \cdot |m_y - 0.5| \cdot (0.5 - |m_y - 0.5|) + \left(\frac{y}{v_2}+1\right)^2\right) \cdot \left(1 + \frac{3z}{v_3}\right) \\ &- \left(\frac{x}{v_1}+1\right)^2 \left(1+\frac{y}{v_2}\right)^2 \left(1+\frac{z}{v_3}\right)^2 \end{aligned} \tag{123}$$

The standard deviation of N_p is the square root of the expression in (123).

2. 2 small dimensions, i.e., one of the following is true:

   $x \geq v_1,\ y < v_2,\ z < v_3$

   $x < v_1,\ y \geq v_2,\ z < v_3$

   $x < v_1,\ y < v_2,\ z \geq v_3$

   Since the three dimensions are interchangeable, we will denote the 2 small dimensions with y and z and the third dimension with x. Thus, $x \geq v_1$, $y < v_2$, $z < v_3$. As shown in Appendix D.3, the number of partially covered voxels in this case is given by

$$N_p = \begin{cases} f_x + 2 & \text{if } m_y \leq q_y \leq 1 - m_y \text{ and } m_z \leq q_z \leq 1 - m_z \\ 2(f_x+2) & \text{if } m_y \leq q_y \leq 1 - m_y \text{ and } (0 \leq q_z < m_z \text{ or } 1 - m_z < q_z \leq 1) \\ 2(f_x+2) & \text{if } m_z \leq q_z \leq 1 - m_z \text{ and } (0 \leq q_y < m_y \text{ or } 1 - m_y < q_y \leq 1) \\ 4(f_x+2) & \text{if } (0 \leq q_y < m_y \text{ or } 1 - m_y < q_y \leq 1) \text{ and } (0 \leq q_z < m_z \text{ or } 1 - m_z < q_z \leq 1) \end{cases} \tag{124}$$

with $q_y = c_y/v_2$ and $q_z = c_z/v_3$, as before. The distribution of N_p is thus given by

$$N_p = \begin{cases} f_x^{(1)} + 2 & \text{with probability } p(f_x^{(1)}) \cdot \left(1 - \frac{y}{v_2}\right) \cdot \left(1 - \frac{z}{v_3}\right) \\ f_x^{(2)} + 2 & \text{with probability } p(f_x^{(2)}) \cdot \left(1 - \frac{y}{v_2}\right) \cdot \left(1 - \frac{z}{v_3}\right) \\ 2(f_x^{(1)} + 2) & \text{with probability } p(f_x^{(1)}) \cdot \left(\frac{y}{v_2} \cdot \left(1 - \frac{z}{v_3}\right) + \frac{z}{v_3}\left(1 - \frac{y}{v_2}\right)\right) \\ 2(f_x^{(2)} + 2) & \text{with probability } p(f_x^{(2)}) \cdot \left(\frac{y}{v_2} \cdot \left(1 - \frac{z}{v_3}\right) + \frac{z}{v_3}\left(1 - \frac{y}{v_2}\right)\right) \\ 4(f_x^{(1)} + 2) & \text{with probability } p(f_x^{(1)}) \cdot \frac{y}{v_2} \cdot \frac{z}{v_3} \\ 4(f_x^{(2)} + 2) & \text{with probability } p(f_x^{(2)}) \cdot \frac{y}{v_2} \cdot \frac{z}{v_3} \end{cases} \tag{125}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22) and $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23).

As shown in Appendix D.3, the expected value of N_p is given by

$$E(N_p) = \left(\frac{x}{v_1} + 1\right)\left(\frac{y}{v_2} + 1\right)\left(\frac{z}{v_3} + 1\right).$$

Note that it is exactly the same as the expected value of the number of partially covered voxels in all cases of 1 small dimension. The variance of N_p is given by

$$\begin{aligned} Var(N_p) \quad &= \quad \left[\left(4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|) + \left(\frac{x}{v_1} + 1\right)^2\right)\right. \\ &\times \quad \left(\frac{3y}{v_2} + 1\right) \\ &\times \quad \left.\left(\frac{3z}{v_3} + 1\right)\right] \\ &- \quad \left(\frac{x}{v_1} + 1\right)^2 \left(\frac{y}{v_2} + 1\right)^2 \left(\frac{z}{v_3} + 1\right)^2. \end{aligned} \tag{126}$$

The standard deviation is the square root of the expression in (126).

3. 3 small dimensions: $x < v_1$, $y < v_2$ and $z < v_3$

As shown in Appendix D.3, the number of partially covered voxels is given by

$$N_p = \begin{cases} 1 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } m_y \le q_y \le 1 - m_y \text{ and } m_z \le q_z \le 1 - m_z \\ 2 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } m_y \le q_y \le 1 - m_y \text{ and } (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \\ 2 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and } m_z \le q_z \le 1 - m_z \\ 2 & \text{if } (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } m_y \le q_y \le 1 - m_y \text{ and } m_z \le q_z \le 1 - m_z \\ 4 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and} \\ & (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \\ 4 & \text{if } (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } m_y \le q_y \le 1 - m_y \text{ and} \\ & (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \\ 4 & \text{if } (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and} \\ & m_z \le q_z \le 1 - m_z \\ 8 & (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and} \\ & (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \end{cases} \tag{127}$$

where $q_x = c_x/v_1$, $q_y = c_y/v_2$ and $q_z = c_z/v_3$, as before. Therefore, as also shown in Appendix D.3, the distribution of the number of partially covered voxels in this case is given by

$$N_p = \begin{cases} 1 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 2 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\left(1 - \frac{y}{v_2}\right)\frac{z}{v_3} + \left(1 - \frac{x}{v_1}\right)\frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) + \frac{x}{v_1}\left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 4 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\frac{y}{v_2}\frac{z}{v_3} + \frac{x}{v_1}\left(1 - \frac{y}{v_2}\right)\frac{z}{v_3} + \frac{x}{v_1}\frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) \\ 8 & \text{with probability } \frac{x}{v_1}\frac{y}{v_2}\frac{z}{v_3}. \end{cases} \tag{128}$$

The expected value of N_p is given by

$$E(N_p) = \left(\frac{x}{v_1} + 1\right)\left(\frac{y}{v_2} + 1\right)\left(\frac{z}{v_3} + 1\right).$$

Note that the expected value is the same as that in the cases of 1 and 2 small dimensions. The variance of N_p is given by

$$Var(N_p) = \left(\frac{3x}{v_1} + 1\right)\left(\frac{3y}{v_2} + 1\right)\left(\frac{3z}{v_3} + 1\right) - \left(\frac{x}{v_1} + 1\right)^2\left(\frac{y}{v_2} + 1\right)^2\left(\frac{z}{v_3} + 1\right)^2. \tag{129}$$

The standard deviation is the square root of the expression in (129).

3.5.2 Fractions Covered in Partially Covered Voxels

1. 1 small dimension, i.e., one of the following is true:

   $x \ge v_1,\ y \ge v_2,\ z < v_3$

   $x \ge v_1,\ y < v_2,\ z \ge v_3$

   $x < v_1,\ y \ge v_2,\ z \ge v_3$

Still denoting the small dimension with z, as before, we have $x \geq v_1$, $y \geq v_2$, $z < v_3$. As discussed in Section 3.5.1, in this case there are either $(f_x+2)(f_y+2)$ or $2(f_x+2)(f_y+2)$ partially covered voxels. The former subcase, referred to as 1(a) at the beginning of Section 3.5, corresponds to just 1 partially covered voxel length along the z dimension. In this case, which occurs with probability $1 - z/v_3$ (as shown in Appendix D.2), the fractions covered in the partially covered voxels are as follows:

- $f_x f_y$ voxels are each covered z/v_3 of the volume,
- f_y voxels are each covered $p_x^{(1)}(z/v_3)$ of the volume,
- f_y voxels are each covered $p_x^{(2)}(z/v_3)$ of the volume,
- f_x voxels are each covered $p_y^{(1)}(z/v_3)$ of the volume,
- f_x voxels are each covered $p_y^{(2)}(z/v_3)$ of the volume,
- 1 voxel covered $p_x^{(1)} p_y^{(1)}(z/v_3)$ of the volume,
- 1 voxel covered $p_x^{(1)} p_y^{(2)}(z/v_3)$ of the volume,
- 1 voxel covered $p_x^{(2)} p_y^{(1)}(z/v_3)$ of the volume,
- 1 voxel covered $p_x^{(2)} p_y^{(2)}(z/v_3)$ of the volume.

Since $x \geq v_1$, each of $p_x^{(1)}$ and $p_x^{(2)}$ has a Uniform(0,1) distribution. Likewise, since $y \geq v_2$, each of $p_y^{(1)}$ and $p_y^{(2)}$ has a Uniform(0,1) distribution, as well. Thus, each of the fractions covered $p_x^{(1)}(z/v_3)$, $p_x^{(2)}(z/v_3)$, $p_y^{(1)}(z/v_3)$, and $p_y^{(2)}(z/v_3)$ has a Uniform(0, z/v_3) distribution, and expected value, median, and standard deviation of $z/(2v_3)$, $z/(2v_3)$, and $z/(\sqrt{12}v_3)$, respectively. The probability of any one of these four fractions covered being between two values a and b with $0 \leq a < b \leq z/v_3$ is equal to $(b-a)v_3/z$.

According to the results in Appendix F.4, each of the products $p_x^{(1)} p_y^{(1)}(z/v_3)$, $p_x^{(1)} p_y^{(2)}(z/v_3)$, $p_x^{(2)} p_y^{(1)}(z/v_3)$ and $p_x^{(2)} p_y^{(2)}(z/v_3)$ has the probability density function given by

$$g_W(w) = \frac{v_3}{z} \cdot (log(z/v_3) - log\ w)\,, \quad 0 \leq w \leq z/v_3 \tag{130}$$

and 0 otherwise. As also shown in that section, the probability that any of these four fractions is between two values A and B with $0 \leq A < B \leq z/v_3$:

$$P(A \leq W \leq B) = \frac{v_3}{z} \cdot (log(z/v_3) + 1) \cdot (B - A) \ + \ \frac{v_3}{z} \cdot (A\ log\ A - B\ log\ B) \tag{131}$$

with $0 \cdot log\ 0 \equiv 0$. The expected value, median and standard deviation of each of these four fractions covered are $z/(4v_3)$, $(0.19z)/v_3$, and $(0.22z)/v_3$, respectively.

The subcase of $2(f_x+2)(f_y+2)$ partially covered voxels (referred to as 1(b) at the beginning of Section 3.5)corresponds to 2 partially covered voxel lengths along the z dimension, one $p_z^{(1)}$ and the other $p_z^{(2)}$ long. In this case, which occurs with probability z/v_3 (as shown in Appendix D.2), the fractions covered in the partially covered voxels are as follows:

- $f_x f_y$ voxels at $p_z^{(1)}$ each,
- $f_x f_y$ voxels at $p_z^{(2)}$ each,
- f_x voxels at $p_y^{(1)} p_z^{(1)}$ each,
- f_x voxels at $p_y^{(1)} p_z^{(2)}$ each,
- f_x voxels at $p_y^{(2)} p_z^{(1)}$ each,
- f_x voxels at $p_y^{(2)} p_z^{(2)}$ each,
- f_y voxels at $p_x^{(1)} p_z^{(1)}$ each,
- f_y voxels at $p_x^{(1)} p_z^{(2)}$ each,
- f_y voxels at $p_x^{(2)} p_z^{(1)}$ each,
- f_y voxels at $p_x^{(2)} p_z^{(2)}$ each,
- 1 voxel at $p_x^{(1)} p_y^{(1)} p_z^{(1)}$,
- 1 voxel at $p_x^{(1)} p_y^{(1)} p_z^{(2)}$,
- 1 voxel at $p_x^{(1)} p_y^{(2)} p_z^{(1)}$,
- 1 voxel at $p_x^{(1)} p_y^{(2)} p_z^{(2)}$,
- 1 voxel at $p_x^{(2)} p_y^{(1)} p_z^{(1)}$,
- 1 voxel at $p_x^{(2)} p_y^{(1)} p_z^{(2)}$,
- 1 voxel at $p_x^{(2)} p_y^{(2)} p_z^{(1)}$,
- 1 voxel at $p_x^{(2)} p_y^{(2)} p_z^{(2)}$.

By analogy to the results obtained in Appendix E.3, each of the fractions $p_z^{(1)}$ and $p_z^{(2)}$ has a Uniform(0, z/v_3) distribution, so their expected value, median and standard deviation are $z/(2v_3)$, $z/(2v_3)$, and $z/(\sqrt{12}v_3)$, respectively. Each fraction's probability of being between values a and b with $0 \leq a < b \leq z/v_3$ is equal to $(b-a)v_3/z$.

Since $x \geq v_1$, $p_x^{(1)}$ and $p_x^{(2)}$ each has a Uniform(0, 1) distribution. Also, $y \geq v_2$, so each of $p_y^{(1)}$ and $p_y^{(2)}$ has a Uniform(0, 1) distribution. Finally, as each of $p_z^{(1)}$ and $p_z^{(2)}$ has a Uniform(0, z/v_3) distribution, and these fractions are independent from their counterparts in the x and y dimensions, according to the results obtained in Appendix F.2, each of the products $p_x^{(1)} p_z^{(1)}$, $p_x^{(1)} p_z^{(2)}$, $p_x^{(2)} p_z^{(1)}$, $p_x^{(2)} p_z^{(2)}$, as well as $p_y^{(1)} p_z^{(1)}$, $p_y^{(1)} p_z^{(2)}$, $p_y^{(2)} p_z^{(1)}$, and $p_y^{(2)} p_z^{(2)}$ has the same distribution as the quantities $p_x^{(1)} p_y^{(1)}(z/v_3)$, $p_x^{(1)} p_y^{(2)}(z/v_3)$, $p_x^{(2)} p_y^{(1)}(z/v_3)$, and $p_x^{(2)} p_y^{(2)}(z/v_3)$, considered earlier. Thus, each of these fractions has the probability density function given in (130). The probability that any one of these fractions is in the interval (A, B) with $0 \leq A < B \leq z/v_3$ is therefore the same as that given in (131). The expected value, median and standard deviation of each of these products is $z/(4v_3)$, $(0.19z)/v_3$ and $(0.22z)/v_3$, respectively.

According to the results in Section F.3, each of the 8 triple products $p_x^{(1)}p_y^{(1)}p_z^{(1)}$, $p_x^{(1)}p_y^{(1)}p_z^{(2)}$, $p_x^{(1)}p_y^{(2)}p_z^{(1)}$, $p_x^{(1)}p_y^{(2)}p_z^{(2)}$, $p_x^{(2)}p_y^{(1)}p_z^{(1)}$, $p_x^{(2)}p_y^{(1)}p_z^{(2)}$, $p_x^{(2)}p_y^{(2)}p_z^{(1)}$, and $p_x^{(2)}p_y^{(2)}p_z^{(2)}$ has the probability density function given by

$$h_W(w) = \frac{v_3}{2z}(log(z/v_3) - log\ w)^2, \quad 0 \leq w \leq z/v_3 \tag{132}$$

and 0 otherwise. The probability that any one of these fractions is in the interval (A, B) with $0 \leq A < B \leq z/v_3$ is therefore equal to

$$P(A \leq W \leq B) = \frac{v_3}{2z}\left[B\left((log\ B - log(z/v_3) - 1)^2 + 1\right) - A\left((log\ A - log(z/v_3) - 1)^2 + 1\right)\right] \tag{133}$$

with $0 \cdot log\ 0 \equiv 0$. According to the results obtained in Section F.3, the expected value, median and standard deviation of each of these products is $z/(8v_3)$, $(0.07z)/v_3$ and $(0.15z)/v_3$, respectively. Table 15 summarizes the fractions covered and the corresponding number of voxels at that fraction for each of the two subcases of 1 small dimension in the 3-dimensional setting. The small dimension is denoted with z, so the case when $x \geq v_1$, $y \geq v_2$ and $z < v_3$.

To obtain the *entire* distribution of fractions covered in all partially covered voxels without fixing the region, we extend the steps outlined in Section 2.3.2 to the three-dimensional setting. Extending equation (72) to the 3-dimensional setting, we have that the probability density function of the fraction covered F in any randomly picked partially covered voxel (across both subcases and all regions within each subcase) is given by

$$g(F) = \sum_R \sum_S \sum_{f_x} \sum_{f_y} g(F \mid R, S, f_x, f_y) \cdot p(R \mid S, f_x, f_y) \cdot p(S) \cdot p(f_x) \cdot p(f_y), \tag{134}$$

where

R denotes the region (1 through 9 for $S = (a)$ and 1 through 18 for $S = (b)$),

S denotes the subcase (a) or (b) in Table 15,

$g(F \mid R, S, f_x, f_y)$ denotes the conditional density of the fraction covered in a randomly picked voxel in a given region R within the given subcase S and the values of f_x and f_y,

$p(R \mid S, f_x, f_y)$ is the relative frequency of the region R for a given subcase S and the values of f_x and f_y,

$p(S)$ denotes the probability of each subcase and is given in Table 15,

$p(f_x)$ is given in (19)–(20), and $p(f_y)$ is given in (31)–(32).

Extending the procedure outlined in Table 6 to the 3-dimensional setting and following Table 15, the procedure for generating from the density in (134) is given in Table 16.

2. 2 small dimensions, i.e., one of the following is true:

Sub-case	Proba-bility	Reg-ion	Fraction Covered				# Voxels at the Fraction Covered			
			Expres-sion	Distri-bution	Exp. Value	Std. Dev.	Exp.	Distri-bution	Exp. Value	Std. Dev.
(a)	$1-\frac{z}{v_3}$	1	$\frac{z}{v_3}$	n/a	$\frac{z}{v_3}$	0	$f_x f_y$	Eq.(110)	Eq.(113)	Sq. root of Eq.(114)
		2	$p_x^{(1)}\frac{z}{v_3}$	U$(0,\frac{z}{v_3})$	$\frac{z}{2v_3}$	$\frac{z}{\sqrt{12}v_3}$	f_y	Eqs.(31)–(32)	$\frac{y}{v_2}-1$	Eq.(40)
		3	$p_x^{(2)}\frac{z}{v_3}$	U$(0,\frac{z}{v_3})$	$\frac{z}{2v_3}$	$\frac{z}{\sqrt{12}v_3}$	f_y	Eqs.(31)–(32)	$\frac{y}{v_2}-1$	Eq.(40)
		4	$p_y^{(1)}\frac{z}{v_3}$	U$(0,\frac{z}{v_3})$	$\frac{z}{2v_3}$	$\frac{z}{\sqrt{12}v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		5	$p_y^{(2)}\frac{z}{v_3}$	U$(0,\frac{z}{v_3})$	$\frac{z}{2v_3}$	$\frac{z}{\sqrt{12}v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		6	$p_x^{(1)}p_y^{(1)}\frac{z}{v_3}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	1	n/a	1	0
		7	$p_x^{(1)}p_y^{(2)}\frac{z}{v_3}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	1	n/a	1	0
		8	$p_x^{(2)}p_y^{(1)}\frac{z}{v_3}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	1	n/a	1	0
		9	$p_x^{(2)}p_y^{(2)}\frac{z}{v_3}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	1	n/a	1	0
(b)	$\frac{z}{v_3}$	1	$p_z^{(1)}$	U$(0,\frac{z}{v_3})$	$\frac{z}{2v_3}$	$\frac{z}{\sqrt{12}v_3}$	$f_x f_y$	Eq.(110)	Eq.(113)	Sq. root of Eq. (114)
		2	$p_z^{(2)}$	U$(0,\frac{z}{v_3})$	$\frac{z}{2v_3}$	$\frac{z}{\sqrt{12}v_3}$	$f_x f_y$	Eq.(110)	Eq.(113)	Sq. root of Eq. (114)
		3	$p_x^{(1)}p_z^{(1)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_y	Eqs.(31)–(32)	$\frac{y}{v_2}-1$	Eq.(40)
		4	$p_x^{(1)}p_z^{(2)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_y	Eqs.(31)–(32)	$\frac{y}{v_2}-1$	Eq.(40)
		5	$p_x^{(2)}p_z^{(1)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_y	Eqs.(31)–(32)	$\frac{y}{v_2}-1$	Eq.(40)
		6	$p_x^{(2)}p_z^{(2)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_y	Eqs.(31)–(32)	$\frac{y}{v_2}-1$	Eq.(40)
		7	$p_y^{(1)}p_z^{(1)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		8	$p_y^{(1)}p_z^{(2)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		9	$p_y^{(2)}p_z^{(1)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		10	$p_y^{(2)}p_z^{(2)}$	Eq.(130)	$\frac{z}{4v_3}$	$\frac{0.22z}{v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		11	$p_x^{(1)}p_y^{(1)}p_z^{(1)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0
		12	$p_x^{(1)}p_y^{(1)}p_z^{(2)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0
		13	$p_x^{(1)}p_y^{(2)}p_z^{(1)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0
		14	$p_x^{(1)}p_y^{(2)}p_z^{(2)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0
		15	$p_x^{(2)}p_y^{(1)}p_z^{(1)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0
		16	$p_x^{(2)}p_y^{(1)}p_z^{(2)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0
		17	$p_x^{(2)}p_y^{(2)}p_z^{(1)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0
		18	$p_x^{(2)}p_y^{(2)}p_z^{(2)}$	Eq.(132)	$\frac{z}{8v_3}$	$\frac{0.15z}{v_3}$	1	n/a	1	0

Table 15: The fraction of the voxel covered for each of the partially covered voxels, its distribution, expected value and standard deviation, along with the number of voxels with that fraction covered, its distribution, expected value and standard deviation, for the case of 1 small dimension in the 3-dimensional setting. The small dimension is denoted with z, so $x \geq v_1$, $y \geq v_2$ and $z < v_3$.

1) Independently generate f_x according to (19)–(20) and f_y according to (31)–(32) (using Uniform(0,1) random variables, as discussed in Appendix C.2).

2) Given f_x and f_y generated in step 1, independently generate a value from the distribution of $p_x^{(1)}$ given f_x and a value from the distribution of $p_y^{(1)}$ given f_y according to (56)–(59).

3) Set $p_x^{(2)} = x/v_1 - f_x - p_x^{(1)}$ and $p_y^{(2)} = y/v_2 - f_y - p_y^{(1)}$

4) Generate the subcase $S =$ (a) or (b) according to their probabilities in Table 15 (using Uniform (0,1) random variables, as discussed in Appendix C.2).

5) If $S = (a)$ in step 4,
   – set F to be a sequence of the following $f_x f_y + 2(f_x + f_y) + 4$ values:

$$\underbrace{z/v_3, \ldots}_{\text{repeat } f_x f_y \text{ times}}, \quad \underbrace{(z/v_3)\cdot p_x^{(1)},\ (z/v_3)\cdot p_x^{(2)}, \ldots,}_{\text{repeat } f_y \text{ times}} \quad \underbrace{(z/v_3)\cdot p_y^{(1)},\ (z/v_3)\cdot p_y^{(2)}, \ldots,}_{\text{repeat } f_x \text{ times}}$$
$$(z/v_3)\cdot p_x^{(1)}p_y^{(1)},\ (z/v_3)\cdot p_x^{(1)}p_y^{(1)},\ (z/v_3)\cdot p_x^{(1)}p_y^{(1)},\ (z/v_3)\cdot p_x^{(1)}p_y^{(1)}$$

   If $S = (b)$ in step 4,
   – generate $p_z^{(1)} \sim U(0, z/v_3)$,
   – set $p_z^{(2)} = z/v_3 - p_z^{(1)}$, and
   – set F to be a sequence of the following $2f_x f_y + 4(f_x + f_y) + 8$ values:

$$\underbrace{p_z^{(1)},\ p_z^{(2)}, \ldots}_{\text{repeat } f_x f_y \text{ times}}, \quad \underbrace{p_x^{(1)}p_z^{(1)},\ p_x^{(1)}p_z^{(2)},\ p_x^{(1)}p_z^{(2)},\ p_x^{(2)}p_z^{(2)}, \ldots,}_{\text{repeat } f_y \text{ times}} \quad \underbrace{p_y^{(1)}p_z^{(1)},\ p_y^{(1)}p_z^{(2)},\ p_y^{(1)}p_z^{(2)},\ p_y^{(2)}p_z^{(2)}, \ldots,}_{\text{repeat } f_x \text{ times}}$$
$$p_x^{(1)}p_y^{(1)}p_z^{(1)},\ p_x^{(1)}p_y^{(1)}p_z^{(2)},\ p_x^{(1)}p_y^{(2)}p_z^{(1)},\ p_x^{(1)}p_y^{(2)}p_z^{(2)},$$
$$p_x^{(2)}p_y^{(1)}p_z^{(1)},\ p_x^{(2)}p_y^{(1)}p_z^{(2)},\ p_x^{(2)}p_y^{(2)}p_z^{(1)},\ p_x^{(2)}p_y^{(2)}p_z^{(2)}$$

6) Repeat steps 1–5 K times for a large value K (e.g., 100000).

   The resulting set of K sequences F will be a sample from the density in (134). This set can then be queried for any summary quantity of interest, such as quantiles, percentiles, means, medians and variances of the fractions covered in all the partially covered voxels.

Table 16: Algorithm for generating values of the fraction covered, F, from the distribution in (134) in the 3-dimensional setting with 1 small dimension (z is assumed to be the small dimension, so $x \geq v_1$, $y \geq v_2$ and $z < v_3$).

$x \geq v_1,\ y < v_2,\ z < v_3$

$x < v_1,\ y \geq v_2,\ z < v_3$

$x < v_1,\ y < v_2,\ z \geq v_3$

As before, we denote the small dimensions with y and z and the third dimension with x. Thus, $x \geq v_1$, $y < v_2$ and $z < v_3$. As discussed in Section 3.5.1, in this case there are either $(f_x + 2)$ (subcase (a), $2(f_x + 2)$ (subcases (b) and (c)) or $4(f_x + 2)$ (subcase (d)) partially covered voxels. The first subcase corresponds to 1 partially covered voxel length along both the y and the z dimensions. In this case, which occurs with probability $(1-y/v_2)(1-z/v_3)$ (as shown in Appendix D.2), the fractions covered in the partially covered voxels are as follows:

- f_x voxels at $yz/(v_2v_3)$ each,
- 1 voxel at $p_x^{(1)} \cdot yz/(v_2v_3)$
- 1 voxel at $p_x^{(2)} \cdot yz/(v_2v_3)$

Since $p_x^{(1)}$ and $p_x^{(2)}$ is each Uniform(0,1), $p_x^{(1)}(yz)/(v_2v_3)$ and $p_x^{(2)}(yz)/(v_2v_3)$ is each Uniform(0, $yz/(v_2v_3)$). Thus, each of these fractions has the expected value, median and standard deviation of $yz/(2v_2v_3)$, $yz/(2v_2v_3)$ and $yz/(\sqrt{12}v_2v_3)$, respectively.

There are $2(f_x + 2)$ partially covered voxels when 1 voxel length is covered along the y dimension, and 2 voxel lengths are covered along the z dimension. This occurs with probability $(1 - y/v_2)(z/v_3)$. The fractions covered in the partially covered voxels in this case are as follows:

- f_x voxels at $p_z^{(1)} \cdot y/v_2$ each
- f_x voxels at $p_z^{(2)} \cdot y/v_2$ each
- 1 voxel at $p_x^{(1)}p_z^{(1)}(y/v_2)$ each
- 1 voxel at $p_x^{(1)}p_z^{(2)}(y/v_2)$ each
- 1 voxel at $p_x^{(2)}p_z^{(1)}(y/v_2)$ each
- 1 voxel at $p_x^{(2)}p_z^{(2)}(y/v_2)$ each

Since $p_z^{(1)}$ and $p_z^{(2)}$ is each Uniform(0,z/v_3), each of the fractions $p_z^{(1)} \cdot y/v_2$ and $p_z^{(2)} \cdot y/v_2$ is Uniform(0, $yz/(v_2v_3)$) and thus has the expected value, median and standard deviation of $yz/(2v_2v_3)$, $yz/(2v_2v_3)$, and $yz/(\sqrt{12}v_2v_3)$, respectively. According to the results obtained in Appendix F.4, each of the fractions $p_x^{(1)}p_z^{(1)}(y/v_2)$, $p_x^{(1)}p_z^{(2)}(y/v_2)$, $p_x^{(2)}p_z^{(1)}(y/v_2)$, and $p_x^{(2)}p_z^{(2)}(y/v_2)$ has the probability density function given by

$$g_W(w) = \frac{v_2v_3}{yz} \cdot \left(log\left(\frac{yz}{v_2v_3}\right) - log\ w \right), \quad 0 \leq w \leq yz/(v_2v_3) \tag{135}$$

and 0 otherwise. Hence, the probability that any of these four fractions is between two values A and B with $0 \leq A < B \leq yz/(v_2 v_3)$:

$$P(A \leq W \leq B) = \frac{v_2 v_3}{yz} \cdot \left(log \left(\frac{yz}{v_2 v_3} \right) + 1 \right) \cdot (B - A) + \frac{v_2 v_3}{yz} \cdot (A \, log \, A - B \, log \, B) \quad (136)$$

with $0 \cdot log \, 0 \equiv 0$. The expected value, median, and standard deviation of each of these four fractions are $yz/(4v_2 v_3)$, $(0.19yz)/(v_2 v_3)$, and $(0.22yz)/(v_2 v_3)$, respectively.

There are also $2(f_x + 2)$ partially covered voxels when 1 voxel length is covered along the z dimension, and 2 voxel lengths are covered along the y dimension. This occurs with probability $(1 - z/v_3)(y/v_2)$. The fractions covered in the partially covered voxels in this case are as follows:

- f_x voxels at $p_y^{(1)} \cdot z/v_3$ each
- f_x voxels at $p_y^{(2)} \cdot z/v_3$ each
- 1 voxel at $p_x^{(1)} p_y^{(1)} (z/v_3)$ each
- 1 voxel at $p_x^{(1)} p_y^{(2)} (z/v_3)$ each
- 1 voxel at $p_x^{(2)} p_y^{(1)} (z/v_3)$ each
- 1 voxel at $p_x^{(2)} p_y^{(2)} (z/v_3)$ each

By symmetry, the distributions, expected values, medians and standard deviations of the above quantities are the same as the ones in the previous paragraph.

Finally, there are $4(f_x + 2)$ voxels when 2 voxel lengths are covered along both the y and the z dimensions. This occurs with probability $yz/(v_2 v_3)$. The fractions covered in the partially covered voxels are as follows:

- f_x voxels at $p_y^{(1)} p_z^{(1)}$ each
- f_x voxels at $p_y^{(1)} p_z^{(2)}$ each
- f_x voxels at $p_y^{(2)} p_z^{(1)}$ each
- f_x voxels at $p_y^{(2)} p_z^{(2)}$ each
- 1 voxel at $p_x^{(1)} p_y^{(1)} p_z^{(1)}$ each
- 1 voxel at $p_x^{(1)} p_y^{(1)} p_z^{(2)}$ each
- 1 voxel at $p_x^{(1)} p_y^{(2)} p_z^{(1)}$ each
- 1 voxel at $p_x^{(1)} p_y^{(2)} p_z^{(2)}$ each
- 1 voxel at $p_x^{(2)} p_y^{(1)} p_z^{(1)}$ each
- 1 voxel at $p_x^{(2)} p_y^{(1)} p_z^{(2)}$ each
- 1 voxel at $p_x^{(2)} p_y^{(2)} p_z^{(1)}$ each

- 1 voxel at $p_x^{(2)}p_y^{(2)}p_z^{(2)}$ each

According to the results derived in Appendix F.2, the probability density function for each of the fractions $p_y^{(1)}p_z^{(1)}$, $p_y^{(1)}p_z^{(2)}$, $p_y^{(2)}p_z^{(1)}$ and $p_y^{(2)}p_z^{(2)}$ has the same distribution as $p_x^{(1)}p_z^{(1)}(y/v_2)$, $p_x^{(1)}p_z^{(2)}(y/v_2)$, $p_x^{(2)}p_z^{(1)}(y/v_2)$, and $p_x^{(2)}p_z^{(2)}(y/v_2)$. Thus, each of these fractions has the same probability density function as that given in (135). The probability that W is in the interval (A, B) with $0 \leq A < B \leq yz/(v_2v_3)$ is therefore the same as that given in (136). The expected value, median and standard deviation of each of these four fractions are thus $yz/(4v_2v_3)$, $(0.19yz)/(v_2v_3)$, and $(0.22yz)/(v_2v_3)$, respectively.

According to the results derived in Section F.3, the probability density function of each of the fractions $p_x^{(1)}p_y^{(1)}p_z^{(1)}$, $p_x^{(1)}p_y^{(1)}p_z^{(2)}$, $p_x^{(1)}p_y^{(2)}p_z^{(1)}$, $p_x^{(1)}p_y^{(2)}p_z^{(2)}$, $p_x^{(2)}p_y^{(1)}p_z^{(1)}$, $p_x^{(2)}p_y^{(1)}p_z^{(2)}$, $p_x^{(2)}p_y^{(2)}p_z^{(1)}$, and $p_x^{(2)}p_y^{(2)}p_z^{(2)}$ is given by

$$g_W(w) = \frac{v_2v_3}{2yz}\left(log\left(\frac{yz}{v_2v_3}\right) - log\ w\right)^2, \quad 0 \leq w \leq \frac{yz}{v_2v_3}. \tag{137}$$

The probability that W is in the interval (A, B) with $0 \leq A < B \leq yz/(v_2v_3)$ is therefore equal to

$$P(A \leq W \leq B) = \frac{v_2v_3}{2yz}B\left((log\ B - log\left(\frac{yz}{v_2v_3}\right) - 1)^2 + 1\right) - A\left((log\ A - log\left(\frac{yz}{v_2v_3}\right) - 1)^2 + 1\right) \tag{138}$$

with $0 \cdot log\ 0 \equiv 0$. The expected value, median and standard deviation of each of these eight fractions are $yz/(8v_2v_3)$, $(0.07yz)/(v_2v_3)$, and $(0.15yz)/(v_2v_3)$, respectively. Table 17 summarizes the fractions covered and the corresponding number of voxels at that fraction for each of the four subcases of the case of 2 small dimensions in the 3-dimensional setting. The small dimensions are denoted with y and z, so when $x \geq v_1$, $y < v_2$ and $z < v_3$.

To obtain the *entire* distribution of fractions covered in all partially covered voxels without fixing the region, we extend the steps outlined for 1 small dimension in the 2-dimensional setting to 2 small dimensions in the 3-dimensional setting. As in (72), we have that the probability density function of the fraction covered F in any randomly picked partially covered voxel (across both subcases and all regions within each subcase) is given by

$$g(F) = \sum_R \sum_S \sum_{f_x} g(F \mid R, S, f_x) \cdot p(R \mid S, f_x) \cdot p(S) \cdot p(f_x), \tag{139}$$

where

R denotes the region (1 through 3 for $S = (a)$ and 1 through 6 for $S = (b)$ or $S = (c)$ and 1 through 12 for $S = (d)$),

S denotes the subcase (a), (b), (c) or (d) in Table 17,

Sub-case	Probability	Region	Fraction Covered				# Voxels at the Fraction Covered			
			Expression	Distribution	Exp. Value	Std. Dev.	Exp.	Distribution	Exp. Value	Std. Dev.
(a)	$\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right)$	1	$\frac{yz}{v_2v_3}$	n/a	$\frac{yz}{v_2v_3}$	0	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		2	$p_x^{(1)}\cdot yz/(v_2v_3)$	$\mathrm{U}(0,\frac{yz}{v_2v_3})$	$\frac{yz}{2v_2v_3}$	$\frac{yz}{\sqrt{12}v_2v_3}$	1	n/a	1	0
		3	$p_x^{(2)}\cdot yz/(v_2v_3)$	$\mathrm{U}(0,\frac{yz}{v_2v_3})$	$\frac{yz}{2v_2v_3}$	$\frac{yz}{\sqrt{12}v_2v_3}$	1	n/a	1	0
(b)	$\left(1-\frac{y}{v_2}\right)\frac{z}{v_3}$	1	$p_z^{(1)}\cdot y/v_2$	$\mathrm{U}(0,\frac{yz}{v_2v_3})$	$\frac{yz}{2v_2v_3}$	$\frac{yz}{\sqrt{12}v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		2	$p_z^{(2)}\cdot y/v_2$	$\mathrm{U}(0,\frac{yz}{v_2v_3})$	$\frac{yz}{2v_2v_3}$	$\frac{yz}{\sqrt{12}v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		3	$p_x^{(1)}p_z^{(1)}(y/v_2)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
		4	$p_x^{(1)}p_z^{(2)}(y/v_2)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
		5	$p_x^{(2)}p_z^{(1)}(y/v_2)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
		6	$p_x^{(2)}p_z^{(2)}(y/v_2)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
(c)	$\left(1-\frac{z}{v_3}\right)\frac{y}{v_2}$	1	$p_y^{(1)}\cdot z/v_3$	$\mathrm{U}(0,\frac{yz}{v_2v_3})$	$\frac{yz}{2v_2v_3}$	$\frac{yz}{\sqrt{12}v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		2	$p_y^{(2)}\cdot z/v_3$	$\mathrm{U}(0,\frac{yz}{v_2v_3})$	$\frac{yz}{2v_2v_3}$	$\frac{yz}{\sqrt{12}v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		3	$p_x^{(1)}p_y^{(1)}(z/v_3)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
		4	$p_x^{(1)}p_y^{(2)}(z/v_3)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
		5	$p_x^{(2)}p_y^{(1)}(z/v_3)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
		6	$p_x^{(2)}p_y^{(2)}(z/v_3)$	Eq.(135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	1	n/a	1	0
(d)	$\frac{yz}{v_2v_3}$	1	$p_y^{(1)}p_z^{(1)}$	Eq. (135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		2	$p_y^{(1)}p_z^{(2)}$	Eq. (135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		3	$p_y^{(2)}p_z^{(1)}$	Eq. (135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		4	$p_y^{(2)}p_z^{(2)}$	Eq. (135)	$\frac{yz}{4v_2v_3}$	$\frac{0.22yz}{v_2v_3}$	f_x	Eqs.(19)–(20)	$\frac{x}{v_1}-1$	Eq.(26)
		5	$p_x^{(1)}p_y^{(1)}p_z^{(1)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0
		6	$p_x^{(1)}p_y^{(1)}p_z^{(2)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0
		7	$p_x^{(1)}p_y^{(2)}p_z^{(1)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0
		8	$p_x^{(1)}p_y^{(2)}p_z^{(2)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0
		9	$p_x^{(2)}p_y^{(1)}p_z^{(1)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0
		10	$p_x^{(2)}p_y^{(1)}p_z^{(2)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0
		11	$p_x^{(2)}p_y^{(2)}p_z^{(1)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0
		12	$p_x^{(2)}p_y^{(2)}p_z^{(2)}$	Eq. (137)	$\frac{yz}{8v_2v_3}$	$\frac{0.15yz}{v_2v_3}$	1	n/a	1	0

Table 17: The fraction of the voxel covered for each of the partially covered voxels, its distribution, expected value and standard deviation, along with the number of voxels with that fraction covered, its distribution, expected value and standard deviation, for the case of 2 small dimensions in the 3-dimensional setting. The small dimensions are denoted with y and z, so $x \geq v_1$, $y < v_2$ and $z < v_3$.

$g(F \mid R, S, f_x)$ denotes the conditional density of the fraction covered in a randomly picked voxel in a given region R within the given subcase S and the value of f_x,

$p(R \mid S, f_x)$ is the relative frequency of the region R for a given subcase S and value of f_x,

$p(S)$ denotes the probability of each subcase and is given in Table 17,

$p(f_x)$ is given in (19)–(20).

Following Table 17 and the steps equivalent to those of Table 6, we have the procedure for generating from the density in (139) as outlined in Table 18.

3. 3 small dimensions, i.e., $x < v_1$, $y < v_2$ and $z < v_3$.
As discussed in Section 3.5.1, in this case there are either 1 (subcase (a)), 2 (subcases (b)–(d)), 4 (subcases (e)–(g)) or 8 (subcase (h)) partially covered voxels. The subcase of 1 partially covered voxel corresponds to 1 partially covered voxel length along each of 3 dimensions. In this case, which occurs with probability $(1-x/v_1)(1-y/v_2)(1-z/v_3)$ (as shown in Appendix D.2), the fraction covered in the 1 partially covered voxel is the entire area of the object scaled to the voxel dimensions, i.e., $xyz/(v_1v_2v_3)$.

The subcases of 2 partially covered voxels corresponds to 1 partially covered voxel lengths along two of the dimensions and 2 partially covered voxel lengths along the third dimension. Let us first consider subcase (b), in which the dimension with 2 partially covered voxel lengths is z. In this case, which occurs with probability $(1-x/v_1)(1-y/v_2)(z/v_3)$ (as shown in Appendix D.2), the fractions covered in the partially covered voxels are as follows:

- 1 voxel at $p_z^{(1)} \cdot xy/(v_1v_2)$
- 1 voxel at $p_z^{(2)} \cdot xy/(v_1v_2)$

Since $p_z^{(1)}$ and $p_z^{(2)}$ is each Uniform$(0, z/v_3)$, each of the fractions covered $p_z^{(1)} \cdot xy/(v_1v_2)$ and $p_z^{(2)} \cdot xy/(v_1v_2)$ has a Uniform$(0,\ xyz/(v_1v_2v_3))$ distribution. The expected value, median and standard deviation of each is thus $xyz/(2v_1v_2v_3)$, $xyz/(2v_1v_2v_3)$ and $xyz/(\sqrt{12}v_1v_2v_3)$, respectively. The subcase (c) is the same as (b), with the y and z quantities interchanged, while subcase (d) is the same as (b), with the x and z quantities interchanged.

The subcases of 4 partially covered voxels corresponds to 2 partially covered voxel lengths along two of the dimensions and 1 partially covered voxel length along the third dimension. Let us first consider subcase (e), in which the dimensions with 2 partially covered voxel lengths are y and z. In this case, which occurs with probability $(1-x/v_1)(y/v_2)(z/v_3)$ (as shown in Appendix D.2), the fractions covered in the partially covered voxels are as follows:

- 1 voxel at $p_y^{(1)}p_z^{(1)}(x/v_1)$
- 1 voxel at $p_y^{(1)}p_z^{(2)}(x/v_1)$

1) Generate f_x according to (19)–(20) (using Uniform(0,1) random variables, as discussed in Appendix C.2).

2) Given f_x generated in step 1, generate $p_x^{(1)}$ from the appropriate distribution specified in (56)–(59).

3) Set $p_x^{(2)} = x/v_1 - f_x - p_x^{(1)}$

4) Generate the subcase $S =$ (a), (b), (c) or (d) according to their probabilities in Table 17 (using Uniform (0,1) random variables, as discussed in Appendix C.2).

5) If $S = (a)$ in step 4,
   – set F to be a sequence of the following $f_x + 2$ values:

$$\frac{yz}{v_2 v_3} \cdot \left(\underbrace{1, \ldots}_{\text{repeat } f_x \text{ times}}, \; p_x^{(1)}, \; p_x^{(2)} \right).$$

   If $S = (b)$ in step 4,
   – generate $p_z^{(1)} \sim U(0, z/v_3)$,
   – set $p_z^{(2)} = z/v_3 - p_z^{(1)}$, and
   – set F to be a sequence of the following $2f_x + 4$ values:

$$\frac{y}{v_2} \cdot \left(\underbrace{p_z^{(1)}, \; p_z^{(2)}, \ldots}_{\text{repeat } f_x \text{ times}}, \; p_x^{(1)} p_z^{(1)}, \; p_x^{(1)} p_z^{(2)}, \; p_x^{(2)} p_z^{(1)}, \; p_x^{(2)} p_z^{(2)} \right).$$

   If $S = (c)$ in step 4,
   – generate $p_y^{(1)} \sim U(0, y/v_2)$,
   – set $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and
   – set F to be a sequence of the following $2f_x + 4$ values:

$$\frac{z}{v_3} \cdot \left(\underbrace{p_y^{(1)}, \; p_y^{(2)}, \ldots}_{\text{repeat } f_x \text{ times}}, \; p_x^{(1)} p_y^{(1)}, \; p_x^{(1)} p_y^{(2)}, \; p_x^{(2)} p_y^{(1)}, \; p_x^{(2)} p_y^{(2)} \right).$$

   If $S = (d)$ in step 4,
   – generate $p_y^{(1)} \sim U(0, y/v_2)$ and $p_z^{(1)} \sim U(0, y/v_3)$,
   – set $p_y^{(2)} = y/v_2 - p_y^{(1)}$ and $p_z^{(2)} = z/v_3 - p_z^{(1)}$,
   – set F to be a sequence of the following $4f_x + 8$ values:

$$\underbrace{p_y^{(1)} p_z^{(1)}, \; p_y^{(1)} p_z^{(2)}, \; p_y^{(2)} p_z^{(1)}, \; p_y^{(2)} p_z^{(2)}, \ldots}_{\text{repeat } f_x \text{ times}}$$
$$p_x^{(1)} p_y^{(1)} p_z^{(1)}, \; p_x^{(1)} p_y^{(1)} p_z^{(2)}, \; p_x^{(1)} p_y^{(2)} p_z^{(1)}, \; p_x^{(1)} p_y^{(2)} p_z^{(2)},$$
$$p_x^{(2)} p_y^{(1)} p_z^{(1)}, \; p_x^{(2)} p_y^{(1)} p_z^{(2)}, \; p_x^{(2)} p_y^{(2)} p_z^{(1)}, \; p_x^{(2)} p_y^{(2)} p_z^{(2)}$$

6) Repeat steps 1–5 K times for a large value K (e.g., 100000).

   The resulting set of K sequences F will be a sample from the density in (139). This set can then be queried for any summary quantity of interest, such as quantiles, percentiles, means, medians and variances of the fractions covered in all the partially covered voxels.

Table 18: Algorithm for generating values of the fraction covered, F, from the distribution in (139) in the 3-dimensional setting with 2 small dimensions (y and z are assumed to be the small dimensions, so $x \geq v_1$, $y < v_2$ and $z < v_3$).

- 1 voxel at $p_y^{(2)} p_z^{(1)} (x/v_1)$
- 1 voxel at $p_y^{(2)} p_z^{(2)} (x/v_1)$

According to the results derived in Appendix F.4, the probability density function for each of the fractions is given by

$$g_W(w) = \frac{v_1 v_2 v_3}{xyz} \cdot \left(log \ \frac{xyz}{v_1 v_2 v_3} - log \ w \right), \quad 0 \leq w \leq xyz/(v_1 v_2 v_3) \tag{140}$$

and 0 otherwise. Therefore, according to (338), the probability that any of these four fractions is in the interval (A, B) with $0 \leq A < B \leq xyz/(v_1 v_2 v_3)$ is equal to

$$P(A \leq W \leq B) = \frac{v_1 v_2 v_3}{xyz} \cdot \left(log \ \frac{xyz}{v_1 v_2 v_3} + 1 \right) \cdot (B - A) \ + \ \frac{v_1 v_2 v_3}{xyz} \cdot (A \ log \ A - B \ log \ B) \tag{141}$$

with $0 \cdot log \ 0 \equiv 0$, as before. From the results developed in Appendix F.4, we have that the expected value, median and standard deviation of each of the four fractions is equal to $xyz/(4v_1 v_2 v_3)$, $(0.19xyz)/(v_1 v_2 v_3)$, and $(0.22xyz)/(v_1 v_2 v_3)$, respectively. The results for subcase (f) are the same with the x and y quantities interchanged, while the results for subcase (g) are the same with the x and z quantities interchanged.

The subcase of 8 partially covered voxels occurs when there are 2 voxel lenghts partially covered along each of the 3 dimensions. In this case, which occurs with probability $xyz/(v_1 v_2 v_3)$, the fractions covered in the partially covered voxels are as follows:

- 1 voxel at $p_x^{(1)} p_y^{(1)} p_z^{(1)}$
- 1 voxel at $p_x^{(1)} p_y^{(1)} p_z^{(2)}$
- 1 voxel at $p_x^{(1)} p_y^{(2)} p_z^{(1)}$
- 1 voxel at $p_x^{(1)} p_y^{(2)} p_z^{(2)}$
- 1 voxel at $p_x^{(2)} p_y^{(1)} p_z^{(1)}$
- 1 voxel at $p_x^{(2)} p_y^{(1)} p_z^{(2)}$
- 1 voxel at $p_x^{(2)} p_y^{(2)} p_z^{(1)}$
- 1 voxel at $p_x^{(2)} p_y^{(2)} p_z^{(2)}$

According to the results developed in Section F.3, each of these 8 fractions has the probability density function given by

$$g_W(w) = \frac{v_1 v_2 v_3}{2xyz} \left(log \ \frac{xyz}{v_1 v_2 v_3} - log \ w \right)^2, \quad 0 \leq w \leq xyz/(v_1 v_2 v_3) \tag{142}$$

and 0 otherwise. Therefore, according to (333), the probability that any of these eight fractions is in the interval (A, B) with $0 \leq A < B \leq xyz/(v_1 v_2 v_3)$ is equal to

$$P(A \leq W \leq B) = \frac{v_1 v_2 v_3}{2xyz} \left[B \left((log \ B - log \ \frac{xyz}{v_1 v_2 v_3} - 1)^2 + 1 \right) - A \left((log \ A - log \ \frac{xyz}{v_1 v_2 v_3} - 1)^2 + 1 \right) \right] \tag{143}$$

with $0 \cdot log\ 0 \equiv 0$. The expected value, median and standard deviation of each of these eight fractions are $xyz/(8v_1v_2v_3)$, $(0.07xyz)/(v_1v_2v_3)$, and $(0.15xyz)/(v_1v_2v_3)$, respectively.

Table 19 summarizes the fractions covered and the corresponding number of voxels at that fraction for each of the 8 subcases of the case of 3 small dimensions in the 3-dimensional setting, i.e., $x <$, $y < v_2$ and $z < v_3$.

To obtain the *entire* distribution of fractions covered in all partially covered voxels without fixing the region, we extend the steps outlined for 2 small dimensions in the 3-dimensional setting to 3 small dimensions in the 3-dimensional setting. As in (77), we have that the probability density function of the fraction covered F in any randomly picked partially covered voxel (across both subcases and all regions within each subcase) is given by

$$g(F) = \sum_R \sum_S g(F \mid R, S) \cdot p(R \mid S) \cdot p(S), \tag{144}$$

where, as before,

R denotes the region (1 for $S = (a)$, 1 or 2 for $S = (b), (c)$ and (d), 1 through 4 for $S = (e), (f)$, and (g), and 1 through 8 for $S = (h)$).

S denotes the subcase (a), (b), (c), (d), (e), (f), (g) or (h) in Table 19,

$g(F \mid R, S)$ denotes the conditional density of the fraction covered in a randomly picked voxel in a given region R within the given subcase S,

$p(R \mid S)$ is the relative frequency of the region R for a given subcase S,

$p(S)$ denotes the probability (relative frequency) of each subcase and is given in Table 19.

Following Table 19 and the steps equivalent to those of Table 9, we have the procedure for generating from the density in (144) as outlined in Table 20.

1) Generate the subcase $S =$ (a), (b), (c), (d), (e), (f), (g) or (h) according to their probabilities in Table 19 (using Uniform (0,1) random variables, as discussed in Appendix C.2).

2) If $S =$ (a) in step 1,
   – set F equal to $\frac{xyz}{v_1v_2v_3}$.

   If $S =$ (b) in step 1,
   – generate $p_z^{(1)} \sim U(0, z/v_3)$,
   – set $p_z^{(2)} = z/v_3 - p_z^{(1)}$, and
   – set F to be the sequence of the following 2 values:
   $\frac{xy}{v_1v_2} \cdot (p_z^{(1)},\ p_z^{(2)})$.

   If $S =$ (c) in step 1,

– generate $p_y^{(1)} \sim U(0, y/v_2)$,
– set $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and
– set F to be the sequence of the following 2 values:
$\frac{xz}{v_1 v_3} \cdot (p_y^{(1)},\ p_y^{(2)})$.

If S = (d) in step 1,
– generate $p_x^{(1)} \sim U(0, x/v_1)$,
– set $p_x^{(2)} = x/v_1 - p_x^{(1)}$, and
– set F to be the sequence of the following 2 values:
$\frac{yz}{v_2 v_3} \cdot (p_x^{(1)},\ p_x^{(2)})$.

If S = (e) in step 1,
– independently generate $p_y^{(1)} \sim U(0, y/v_2)$ and $p_z^{(1)} \sim U(0, z/v_3)$,
– set $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and $p_z^{(2)} = z/v_3 - p_z^{(1)}$, and
– set F to be the sequence of the following 4 values:
$\frac{x}{v_1} \cdot (p_y^{(1)} p_z^{(1)},\ \ p_y^{(1)} p_z^{(2)},\ \ p_y^{(2)} p_z^{(1)},\ \ p_y^{(2)} p_z^{(2)})$.

If S = (f) in step 1,
– independently generate $p_x^{(1)} \sim U(0, x/v_1)$ and $p_z^{(1)} \sim U(0, z/v_3)$,
– set $p_x^{(2)} = x/v_1 - p_x^{(1)}$, and $p_z^{(2)} = z/v_3 - p_z^{(1)}$, and
– set F to be the sequence of the following 4 values:
$\frac{y}{v_2} \cdot (p_x^{(1)} p_z^{(1)},\ \ p_x^{(1)} p_z^{(2)},\ \ p_x^{(2)} p_z^{(1)},\ \ p_x^{(2)} p_z^{(2)})$.

If S = (g) in step 1,
– independently generate $p_x^{(1)} \sim U(0, x/v_1)$ and $p_y^{(1)} \sim U(0, y/v_2)$,
– set $p_x^{(2)} = x/v_1 - p_x^{(1)}$, and $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and
– set F to be the sequence of the following 4 values:
$\frac{z}{v_3} \cdot (p_x^{(1)} p_y^{(1)},\ \ p_x^{(1)} p_y^{(2)},\ \ p_x^{(2)} p_y^{(1)},\ \ p_x^{(2)} p_y^{(2)})$.

If S = (h) in step 1,
– independently generate $p_x^{(1)} \sim U(0, x/v_1)$, $p_y^{(1)} \sim U(0, y/v_2)$ and $p_z^{(1)} \sim U(0, z/v_3)$,
– set $p_x^{(2)} = x/v_1 - p_x^{(1)}$, $p_y^{(2)} = y/v_2 - p_y^{(1)}$, and $p_z^{(2)} = z/v_3 - p_z^{(1)}$
– set F to be the sequence of the following 8 values:
$p_x^{(1)} p_y^{(1)} p_z^{(1)},\ p_x^{(1)} p_y^{(1)} p_z^{(2)},\ p_x^{(1)} p_y^{(2)} p_z^{(1)},\ p_x^{(1)} p_y^{(2)} p_z^{(2)},$
$p_x^{(2)} p_y^{(1)} p_z^{(1)},\ p_x^{(2)} p_y^{(1)} p_z^{(2)},\ p_x^{(2)} p_y^{(2)} p_z^{(1)},\ p_x^{(2)} p_y^{(2)} p_z^{(2)}$.

3) Repeat steps 1–2 K times for a large value K (e.g., 100000).

The resulting set of K sequences F will be a sample from the density in (144). This set can then be queried for any summary quantity of interest, such as quantiles, percentiles, means, medians

and variances of the fractions covered in all the partially covered voxels.

Table 20: Algorithm for generating values of the fraction covered, F, from the distribution in (144) in the 3-dimensional setting with 3 small dimensions ($x < v_1$, $y < v_2$ and $z < v_3$).

3.6 Summary

Table 21 contains the summary of the resulting variables of interest when a rectangular 3-dimensional object with edge lengths x, y, and z is placed inside a rectangular grid of voxels, each with edge lengths v_1, v_2, and v_3, with $x \geq v_1$, $y \geq v_2$, and $z \geq v_3$ (i.e., no small dimensions) and with the center of the object randomly located inside a voxel at a point (c_x, c_y, c_z) (with $0 \leq c_x \leq v_1$, $0 \leq c_y \leq v_2$, and $0 \leq c_z \leq v_3$). Tables 22 – 24 contain the summaries equivalent to that in Table 21 for the cases of 1 small dimension (exactly one of the following is true: $x < v_1$, $y < v_2$, or $z < v_3$), 2 small dimensions (exactly two of the following is true: $x < v_1$, $y < v_2$, or $z < v_3$), and 3 small dimensions ($x < v_1$, $y < v_2$, and $z < v_3$).

References

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[2] R. M. Kessler, J. R. Ellis, and Eden M. Jr. Analysis of emission tomographic scan data: limitations imposed by resolution and background. *Journal of Computer Assisted Tomography*, 8:514–522, 1984.

[3] G. Casella and R. Berger. *Statistical Inference*. Duxbury Press, Belmont, California, 1990.

Sub-case	Proba-bility	Reg-ion	Fraction Covered				# Voxels at the Fraction Covered
			Expres-sion	Distri-bution	Exp. Value	Std. Dev.	
(a)	$\left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right)$	1	$xyz/(v_1v_2v_3)$	n/a	$\frac{xyz}{v_1v_2v_3}$	0	1
(b)	$\left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\left(\frac{z}{v_3}\right)$	1	$p_z^{(1)}\cdot xy/(v_1v_2)$	$\mathrm{U}(0,\frac{xyz}{v_1v_2v_3})$	$\frac{xyz}{2v_1v_2v_3}$	$\frac{xyz}{\sqrt{12}v_1v_2v_3}$	1
		2	$p_z^{(2)}\cdot xy/(v_1v_2)$	$\mathrm{U}(0,\frac{xyz}{v_1v_2v_3})$	$\frac{xyz}{2v_1v_2v_3}$	$\frac{xyz}{\sqrt{12}v_1v_2v_3}$	1
(c)	$\left(1-\frac{x}{v_1}\right)\left(\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right)$	1	$p_y^{(1)}\cdot xz/(v_1v_3)$	$\mathrm{U}(0,\frac{xyz}{v_1v_2v_3})$	$\frac{xyz}{2v_1v_2v_3}$	$\frac{xyz}{\sqrt{12}v_1v_2v_3}$	1
		2	$p_y^{(2)}\cdot xz/(v_1v_3)$	$\mathrm{U}(0,\frac{xyz}{v_1v_2v_3})$	$\frac{xyz}{2v_1v_2v_3}$	$\frac{xyz}{\sqrt{12}v_1v_2v_3}$	1
(d)	$\left(\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right)$	1	$p_x^{(1)}\cdot yz/(v_2v_3)$	$\mathrm{U}(0,\frac{xyz}{v_1v_2v_3})$	$\frac{xyz}{2v_1v_2v_3}$	$\frac{xyz}{\sqrt{12}v_1v_2v_3}$	1
		2	$p_x^{(2)}\cdot yz/(v_2v_3)$	$\mathrm{U}(0,\frac{xyz}{v_1v_2v_3})$	$\frac{xyz}{2v_1v_2v_3}$	$\frac{xyz}{\sqrt{12}v_1v_2v_3}$	1
(e)	$\left(1-\frac{x}{v_1}\right)\left(\frac{y}{v_2}\right)\left(\frac{z}{v_3}\right)$	1	$p_y^{(1)}p_z^{(1)}(x/v_1)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		2	$p_y^{(1)}p_z^{(2)}(x/v_1)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		3	$p_y^{(2)}p_z^{(1)}(x/v_1)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		4	$p_y^{(2)}p_z^{(2)}(x/v_1)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
(f)	$\left(\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\left(\frac{z}{v_3}\right)$	1	$p_x^{(1)}p_z^{(1)}(y/v_2)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		2	$p_x^{(1)}p_z^{(2)}(y/v_2)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		3	$p_x^{(2)}p_z^{(1)}(y/v_2)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		4	$p_x^{(2)}p_z^{(2)}(y/v_2)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
(g)	$\left(\frac{x}{v_1}\right)\left(\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right)$	1	$p_x^{(1)}p_y^{(1)}(z/v_3)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		2	$p_x^{(1)}p_y^{(2)}(z/v_3)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		3	$p_x^{(2)}p_y^{(1)}(z/v_3)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
		4	$p_x^{(2)}p_y^{(2)}(z/v_3)$	Eq. (140)	$\frac{xyz}{4v_1v_2v_3}$	$\frac{0.22xyz}{v_1v_2v_3}$	1
(h)	$\left(\frac{x}{v_1}\right)\left(\frac{y}{v_2}\right)\left(\frac{z}{v_3}\right)$	1	$p_x^{(1)}p_y^{(1)}p_z^{(1)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1
		2	$p_x^{(1)}p_y^{(1)}p_z^{(2)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1
		3	$p_x^{(1)}p_y^{(2)}p_z^{(1)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1
		4	$p_x^{(1)}p_y^{(2)}p_z^{(2)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1
		5	$p_x^{(2)}p_y^{(1)}p_z^{(1)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1
		6	$p_x^{(2)}p_y^{(1)}p_z^{(2)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1
		7	$p_x^{(2)}p_y^{(2)}p_z^{(1)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1
		8	$p_x^{(2)}p_y^{(2)}p_z^{(2)}$	Eq. (142)	$\frac{xyz}{8v_1v_2v_3}$	$\frac{0.15xyz}{v_1v_2v_3}$	1

Table 19: The fraction of the voxel covered for each of the partially covered voxels, its distribution, expected value and standard deviation, along with the number of voxels with that fraction covered, for the case of 3 small dimensions in the 3-dimensional setting, i.e., when $x < v_1$, $y < v_2$ and $z < v_3$.

Variable	**Notation**	**Expression**	**Distribution**	**Expected Value**	**Standard Deviation**
# Fully Covered Voxels	N_f	Eq. (93)	Eq. (94)	Eq. (95)	Sq. root of (96)
# Partially Covered Voxels	N_p	Eq. (97)	Eq. (98)	Eq. (99)	Sq. root of (100)
Fractions Covered In Partially Covered Voxels	see Tables 13 and 14				

Table 21: Summary of the variables of interest in the 3-dimensional setting for the case when $\boldsymbol{x \geq v_1, y \geq v_2, z \geq v_3}$.

Variable	**Notation**	**Expression**	**Distribution**	**Expected Value**	**Standard Deviation**
# Fully Covered Voxels	N_f	0	fixed at 0	0	0
# Partially Covered Voxels	N_p	Eq. (120)	Eq. (121)	Eq. (122)	Sq. root of (123)
Fractions Covered In Partially Covered Voxels	see Tables 15 and 16				

Table 22: Summary of the variables of interest in the 3-dimensional setting for the case of 1 small dimension. The small dimension is denoted with z, so when $x \geq v_1$, $y \geq v_2$ and $z < v_3$.

Variable	**Notation**	**Expression**	**Distribution**	**Expected Value**	**Standard Deviation**
# Fully Covered Voxels	N_f	0	fixed at 0	0	0
# Partially Covered Voxels	N_p	Eq. (124)	Eq. (125)	Eq. (122)	Sq. root of (126)
Fractions Covered In Partially Covered Voxels	see Tables 17 and 18				

Table 23: Summary of the variables of interest in the 3-dimensional setting for the case of 2 small dimensions, with the small dimensions denoted with y and z, i.e., $x \geq v_1$, $y < v_2$ and $z < v_3$.

Variable	**Notation**	**Expression**	**Distribution**	**Expected Value**	**Standard Deviation**
# Fully Covered Voxels	N_f	0	fixed at 0	0	0
# Partially Covered Voxels	N_p	Eq. (127)	Eq. (128)	Eq. (122)	Sq. root of (129)
Fractions Covered In Partially Covered Voxels	see Tables 19 and 20				

Table 24: Summary of the variables of interest in the 3-dimensional setting for the case of 3 small dimensions, i.e., $x < v_1$, $y < v_2$ and $z < v_3$.

Appendix A Expectation and Variance of f_x

From (19) and (20), we have that

$$\begin{aligned} &\text{if} \quad m_x < 0.5, \\ f_x &= \begin{cases} 2k_x & \text{with probability } 2m_x \\ 2k_x - 1 & \text{with probability } 1 - 2m_x \end{cases} \end{aligned} \tag{145}$$

while

$$\begin{aligned} &\text{if} \quad m_x \geq 0.5, \\ f_x &= \begin{cases} 2k_x & \text{with probability } 2(1 - m_x) \\ 2k_x + 1 & \text{with probability } 2m_x - 1 \end{cases} \end{aligned} \tag{146}$$

Therefore, in the case when $m_x < 0.5$, the expected value of f_x is equal to

$$\begin{aligned} E(f_x) &= 2k_x 2m_x + (2k_x - 1)(1 - 2m_x) \\ &= 4k_x m_x + 2k_x - 1 - 4k_x m_x + 2m_x \\ &= 2(k_x + m_x) - 1 = 2 \cdot \frac{x/2}{v_1} - 1 \\ &= \frac{x}{v_1} - 1 \end{aligned} \tag{147}$$

(By definition of k_x and m_x in Section 2.1, $k_x + m_x = \frac{x/2}{v_1}$.)

In the case when $m_x \geq 0.5$, the expected value of f_x is equal to

$$\begin{aligned} E(f_x) &= 2k_x 2(1 - m_x) + (2k_x + 1)(2m_x - 1) \\ &= 4k_x - 4k_x m_x + 4k_x m_x + 2m_x - 2k_x - 1 \\ &= 2(k_x + m_x) - 1 = 2 \cdot \frac{x/2}{v_1} - 1 \\ &= \frac{x}{v_1} - 1 \end{aligned} \tag{148}$$

Thus, for any value of m_x, $E(f_x) = x/v_1 - 1$.

For any value of m_x, the distribution of f_x is a two-point distribution. For any random variable R that has a two-point distribution given by

$$R = \begin{cases} a & \text{with probability } p \\ b & \text{with probability } 1 - p \end{cases}, \tag{149}$$

the variance of R is given by

$$\begin{aligned} Var(R) &= E(R^2) - (E(R))^2 = a^2 p + b^2(1 - p) - (ap + b(1 - p))^2 \\ &= a^2 p + b^2(1 - p) - a^2 p^2 - 2abp(1 - p) - b^2(1 - p)^2 \\ &= a^2 p(1 - p) + b^2(1 - p)(1 - (1 - p)) - 2abp(1 - p) \\ &= a^2 p(1 - p) + b^2 p(1 - p) - 2abp(1 - p) = p(1 - p)(a - b)^2 \end{aligned} \tag{150}$$

That is, the variance of a random variable with a two-point distribution is equal to the product of the probabilities of the two values times the square of the difference of the two values that the variable takes on.

Therefore, when $m_x < 0.5$, the variance of f_x is equal to

$$Var(f_x) = 2m_x(1 - 2m_x)(2k_x - (2k_x - 1))^2 = 2m_x(1 - 2m_x) \tag{151}$$

and when $m_x \geq 0.5$, the variance of the number of fully covered voxels is given by

$$Var(f_x) = 2(1 - m_x)(2m_x - 1)(2k_x + 1 - 2k_x)^2 = 2(1 - m_x)(2m_x - 1). \tag{152}$$

An equivalent and more compact way to express the two cases above for any value of m_x between 0 and 1 is

$$Var(f_x) = 4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|). \tag{153}$$

Appendix B Conditional Distributions of $p_x^{(i)}$ given f_x and $p_y^{(j)}$ given f_y

We first obtain the conditional distribution of $p_x^{(1)}$ given f_x. Recall from Section 2.1 that $p_x^{(1)}$ is defined as

$$p_x^{(1)} = \left\lceil \frac{c_x - x/2}{v_1} \right\rceil - \frac{c_x - x/2}{v_1} = \left\lceil q_x - \frac{x/2}{v_1} \right\rceil - \left(q_x - \frac{x/2}{v_1} \right), \tag{154}$$

which can be re-written as

$$p_x^{(1)} = b_x - (q_x - h) = b_x - q_x + h \tag{155}$$

where

$$h = \frac{x/2}{v_1}, \tag{156}$$

$$q_x = \frac{c_x}{v_1}, \tag{157}$$

and

$$b_x = \lceil q_x - h \rceil . \tag{158}$$

Thus, $k_x = \lfloor h \rfloor$ and $m_x = h - k_x$. As before, let

$$a_x = \lfloor q_x + h \rfloor . \tag{159}$$

Consider first the case when $m_x < 0.5$. Then f_x can take on two values: $2k_x - 1$ and $2k_x$. Consider first $f_x = 2k_x - 1$, which occurs with probability $1 - 2m_x$ (as shown in Section 2.1). Thus, for a real number n, the conditional distribution function of $p_x^{(1)}$ given $f_x = 2k_x - 1$ is equal to

$$\begin{aligned} P(p_x^{(1)} \leq n \mid f_x = 2k_x - 1) &= P(p_x^{(1)} \leq n \cap f_x = 2k_x - 1)/P(f_x = 2k_x - 1) \\ &= P(p_x^{(1)} \leq n \cap f_x = 2k_x - 1)/(1 - 2m_x) \end{aligned} \tag{160}$$

Now, as discussed in Section 2.1, $f_x = 2k_x - 1$ when $a_x = k_x$ and $b_x = -k_x + 1$. Putting this together with the definitions of a_x and b_x in (159) and (158), the numerator in (160) is

equal to

$$
\begin{aligned}
& P(p_x^{(1)} \le n \ \cap \ f_x = 2k_x - 1) = \\
= \ & P(b_x - q_x + h \le n \ \cap \ a_x = k_x \ \cap \ b_x = -k_x + 1) = \\
= \ & P(-k_x + 1 - q_x + h \le n \ \cap k_x \le q_x + h < k_x + 1 \ \cap \ -k_x < q_x - h \le -k_x + 1) = \\
= \ & P((h - k_x) + 1 - n \le q_x \ \cap \ k_x - h \le q_x < k_x - h + 1 \ \cap \ h - k_x < q_x \le h - k_x + 1) = \\
= \ & P(m_x + 1 - n \le q_x \ \cap \ \underbrace{-m_x \le q_x < 1 - m_x \ \cap \ m_x < q_x \le 1 + m_x}_{m_x > 0 \ \Rightarrow \ -m_x < m_x \text{ and } 1 - m_x < 1 + m_x}) = \\
= \ & P(1 + m_x - n \le q_x \ \cap \ \underbrace{m_x \le q_x < 1 - m_x}_{m_x < 1 - m_x \text{ since } m_x < 0.5}) = \\
= \ & P(\max(1 + m_x - n, m_x) \le q_x < 1 - m_x) \\
= \ & \begin{cases} P(1 + m_x - n \le q_x < 1 - m_x) & \text{if } 1 + m_x - n \ge m_x \Rightarrow n \le 1 \\ P(m_x \le q_x < 1 - m_x) & \text{if } 1 + m_x - n < m_x \Rightarrow n > 1 \end{cases}
\end{aligned} \tag{161}
$$

As shown in Section 2.1, $q_x \sim U(0, 1)$. Thus, for the case $n > 1$, since $0 \le m_x \le 1 - m_x \le 1$ (the 2nd inequality is true since $m_x < 0.5$),

$$
P(m_x \le q_x < 1 - m_x) = 1 - m_x - m_x = 1 - 2m_x. \tag{162}
$$

When $n \le 1$, $1 + m_x - n \ge 0$ (and $1 - m_x \le 1$, as before), so

$$
\begin{aligned}
P(1 + m_x - n \le q_x < 1 - m_x) \ &= \ \begin{cases} 1 - m_x - (1 + m_x - n) & 1 + m_x - n < 1 - m_x \\ 0 & 1 + m_x - n \ge 1 - m_x \end{cases} \\
&= \ \begin{cases} n - 2m_x & n \ge 2m_x \\ 0 & n < 2m_x \end{cases}
\end{aligned} \tag{163}
$$

Thus, putting together (161)–(163),

$$
P(p_x^{(1)} \le n \ \cap \ f_x = 2k_x - 1) = \begin{cases} 0 & n < 2m_x \\ n - 2m_x & 2m_x \le n \le 1 \\ 1 - 2m_x & n > 1 \end{cases} \tag{164}
$$

Hence,

$$
\begin{aligned}
P(p_x^{(1)} \le n \mid f_x = 2k_x - 1) \ &= \ P(p_x^{(1)} \le n \ \cap \ f_x = 2k_x - 1)/P(f_x = 2k_x - 1) \\
&= \ P(p_x^{(1)} \le n \ \cap \ f_x = 2k_x - 1)/(1 - 2m_x) \\
&= \ \begin{cases} 0 & n < 2m_x \\ \frac{n - 2m_x}{1 - 2m_x} & 2m_x \le n \le 1 \\ 1 & n > 1 \end{cases}
\end{aligned} \tag{165}
$$

The expression in (165) is the distribution function of a Uniform$(2m_x, 1)$ random variable, so we have shown that when $m_x < 0.5$,

$$
p_x^{(1)} | \ f_x = 2k_x - 1 \ \sim \ \text{Uniform}(2m_x, 1). \tag{166}
$$

Now, we turn to the conditional distribution of $p_x^{(1)}$ given $f_x = 2k_x$ ($m_x < 0.5$ still). As shown in Section 2.1, $P(f_x = 2k_x) = 2m_x$, so we have

$$\begin{aligned} & P(p_x^{(1)} \leq n \mid f_x = 2k_x) = P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x)/P(f_x = 2k_x) \\ = \ & P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x)/(2m_x). \end{aligned} \tag{167}$$

We can derive $P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x)$ using an argument equivalent to that used above to derive $P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x - 1)$, but it is easier to note that when $m_x < 0.5$, the event that $f_x = 2k_x$ is the complement of the event that $f_x = 2k_x - 1$, so

$$P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x) = P(p_x^{(1)} \leq n) - P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x - 1) \tag{168}$$

Now, since, as shown in Section 2.1, $p_x^{(1)} \sim U(0,1)$, we have

$$P(p_x^{(1)} \leq n) = \begin{cases} 0 & n < 0 \\ n & 0 \leq n \leq 1 \\ 1 & n > 1 \end{cases} \tag{169}$$

so putting (169) together with (164), we have

$$P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x) = \begin{cases} 0 - 0 & n < 0 \\ n - 0 & 0 \leq n \leq 2m_x \\ n - (n - 2m_x) & 2m_x < n \leq 1 \\ 1 - (1 - 2m_x) & n > 1 \end{cases} \tag{170}$$

$$= \begin{cases} 0 & n < 0 \\ n & 0 \leq n \leq 2m_x \\ 2m_x & n > 2m_x \end{cases} \tag{171}$$

Thus,

$$\begin{aligned} & P(p_x^{(1)} \leq n \mid f_x = 2k_x) = P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x)/(2m_x) \\ = \ & \begin{cases} 0 & n < 0 \\ \frac{n}{2m_x} & 0 \leq n \leq 2m_x \\ 1 & n > 2m_x \end{cases} \end{aligned} \tag{172}$$

The above is the distribution function of a Uniform(0, $2m_x$) random variable, so have shown that when $m_x < 0.5$,

$$p_x^{(1)} | \ f_x = 2k_x \ \sim \ \text{Uniform}(0, 2m_x). \tag{173}$$

We derive the conditional distribution of $p_x^{(1)}$ given f_x for the case $m_x \geq 0.5$ by repeating the arguments used above for the case $m_x < 0.5$. When $m_x \geq 0.5$, f_x can take on two values: $2k_x + 1$ and $2k_x$. Consider first $f_x = 2k_x + 1$, which occurs with probability $2m_x - 1$ (as shown in Section 2.1). Thus, for a real number n, the conditional distribution function of $p_x^{(1)}$ given $f_x = 2k_x + 1$ is equal to

$$\begin{aligned} P(p_x^{(1)} \leq n \mid f_x = 2k_x + 1) &= P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x + 1)/P(f_x = 2k_x + 1) \\ &= P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x + 1)/(2m_x - 1). \end{aligned} \tag{174}$$

Now, as discussed in Section 2.1, $f_x = 2k_x + 1$ when $a_x = k_x + 1$ and $b_x = -k_x$. Putting this together with the definitions of a_x and b_x in (159) and (158), the numerator in (174) is equal to

$$\begin{aligned}
& P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x + 1) = \\
= \ & P(b_x - q_x + h \leq n \ \cap \ a_x = k_x + 1 \ \cap \ b_x = -k_x) = \\
= \ & P(-k_x - q_x + h \leq n \ \cap k_x + 1 \leq q_x + h < k_x + 2 \ \cap \ -k_x - 1 < q_x - h \leq -k_x) = \\
= \ & P((h - k_x) - n \leq q_x \ \cap \ k_x - h + 1 \leq q_x < k_x - h + 2 \ \cap \ h - k_x - 1 < q_x \leq h - k_x) = \\
= \ & P(m_x - n \leq q_x \ \cap \ \underbrace{1 - m_x \leq q_x < 2 - m_x \ \cap \ m_x - 1 < q_x \leq m_x}_{m_x<1 \ \Rightarrow \ m_x-1<1-m_x \text{ and } m_x<2-m_x}) = \\
= \ & P(m_x - n \leq q_x \ \cap \ \underbrace{1 - m_x \leq q_x < m_x}_{1-m_x<m_x \text{ since } m_x\geq 0.5}) = \\
= \ & P(\max(m_x - n, 1 - m_x) \leq q_x < m_x) \\
= \ & \begin{cases} P(m_x - n \leq q_x < m_x) & m_x - n \geq 1 - m_x \\ P(1 - m_x \leq q_x < m_x) & m_x - n < 1 - m_x \end{cases} \\
= \ & \begin{cases} P(m_x - n \leq q_x < m_x) & n \leq 2m_x - 1 \\ P(1 - m_x \leq q_x < m_x) & n > 2m_x - 1 \end{cases} \\
= \ & \begin{cases} 0 & n < 0 \\ n & 0 \leq n < 2m_x - 1 \\ 2m_x - 1 & n > 2m_x - 1 \end{cases}
\end{aligned} \tag{175}$$

since $q_x \sim U(0, 1)$ (the top 0 is due to the fact that if $n < 0$, $m_x - n > m_x$, so $P(m_x - n \leq q_x < m_x) = 0$). Hence,

$$\begin{aligned}
& P(p_x^{(1)} \leq n \mid f_x = 2k_x + 1) = P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x + 1)/P(f_x = k_x + 1) \\
= \ & \begin{cases} 0 & n < 0 \\ \frac{n}{2m_x-1} & 0 \leq n \leq 2m_x - 1 \\ 1 & n > 2m_x - 1 \end{cases}
\end{aligned} \tag{176}$$

The expression in (176) is the distribution function of a Uniform$(0, 2m_x-1)$ random variable, so we have shown that when $m_x \geq 0.5$,

$$p_x^{(1)} | \ f_x = 2k_x + 1 \ \sim \ \text{Uniform}(0, 2m_x - 1). \tag{177}$$

Now, we derive the distribution of $p_x^{(1)}$ given $f_x = 2k_x$ (for $m_x \geq 0.5$). As shown in Section 2.1, $P(f_x = 2k_x) = 2(1 - m_x)$, so we have

$$\begin{aligned}
& P(p_x^{(1)} \leq n \mid f_x = 2k_x) = P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x)/P(f_x = 2k_x) \\
= \ & P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x)/(2(1 - m_x)).
\end{aligned} \tag{178}$$

Furthermore, we note that when $m_x \geq 0.5$, the event that $f_x = 2k_x$ is the complement of the event that $f_x = 2k_x + 1$, so

$$P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x) = P(p_x^{(1)} \leq n) - P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x + 1), \tag{179}$$

so putting togehter (169) together and (175), we have

$$P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x) = \begin{cases} 0-0 & n<0 \\ n-n & 0 \leq n < 2m_x - 1 \\ n-(2m_x-1) & 2m_x - 1 \leq n \leq 1 \\ 1-(2m_x-1) & n>1 \end{cases} \tag{180}$$

$$= \begin{cases} 0 & n < 2m_x - 1 \\ n-(2m_x-1) & 2m_x - 1 \leq n \leq 1 \\ 2(1-m_x) & n>1 \end{cases} \tag{181}$$

Thus,

$$P(p_x^{(1)} \leq n \mid f_x = 2k_x) = P(p_x^{(1)} \leq n \ \cap \ f_x = 2k_x)/(2(1-m_x))$$
$$= \begin{cases} 0 & n < 2m_x - 1 \\ \frac{n-(2m_x-1)}{2(1-m_x)} & 2m_x - 1 \leq n \leq 1 \\ 1 & n>1 \end{cases} \tag{182}$$

The above is the distribution function of a Uniform$(2m_x - 1, 1)$ random variable, so have shown that when $m_x \geq 0.5$,

$$p_x^{(1)} | \ f_x = 2k_x \ \sim \ \text{Uniform}(2m_x - 1, 1). \tag{183}$$

Now, we derive the conditional distributions of $p_x^{(2)}$ given f_x. We first note that since, as shown in Section 2.1,

$$p_x^{(2)} = \frac{x}{v_1} - f_x - p_x^{(1)}, \tag{184}$$

the conditional distribution function of $p_x^{(2)}$ given f_x (for either range of m_x or value of f_x) is equal to

$$\begin{aligned} P(p_x^{(2)} \leq n \mid f_x) &= P(x/v_1 - f_x - p_x^{(1)} \leq n \mid f_x) \\ &= P(p_x^{(1)} \geq x/v_1 - f_x - n \mid f_x) \\ &= 1 - P(p_x^{(1)} \leq x/v_1 - f_x - n \mid f_x). \end{aligned} \tag{185}$$

Therefore, in each case, the conditional distribution of $p_x^{(1)}$ given f_x derived above can be used to derive the conditional distribution of $p_x^{(2)}$ given f_x. Recall also that by definition of k_x and m_x in Section 2.1,

$$k_x + m_x = \frac{x/2}{v_1}, \tag{186}$$

so

$$2k_x + 2m_x = \frac{x}{v_1}. \tag{187}$$

Then when $m_x < 0.5$, from (185), using the result in (165), we have

$$
\begin{aligned}
& P(p_x^{(2)} \leq n \mid f_x = 2k_x - 1) = 1 - P(p_x^{(1)} \leq x/v_1 - 2k_x + 1 - n \mid f_x = 2k_x - 1) \\
= \; & 1 - \begin{cases} 0 & x/v_1 - 2k_x + 1 - n < 2m_x \\ \frac{x/v_1 - 2k_x + 1 - n - 2m_x}{1 - 2m_x} & 2m_x \leq x/v_1 - 2k_x + 1 - n \leq 1 \\ 1 & x/v_1 - 2k_x + 1 - n > 1 \end{cases} \\
= \; & 1 - \begin{cases} 0 & n > x/v_1 - (2k_x + 2m_x) + 1 \\ \frac{x/v_1 - (2k_x + 2m_x) + 1 - n}{1 - 2m_x} & x/v_1 - 2k_x \leq n \leq x/v_1 - (2k_x + 2m_x) + 1 \\ 1 & n < x/v_1 - 2k_x \end{cases} \\
= \; & 1 - \begin{cases} 0 & n > 1 \\ \frac{1-n}{1-2m_x} & 2m_x \leq n \leq 1 \\ 1 & n < 2m_x \end{cases} \quad \begin{matrix} \text{(due} \\ \text{to} \\ \text{(187))} \end{matrix} \\
= \; & \begin{cases} 0 & n < 2m_x \\ \frac{n - 2m_x}{1 - 2m_x} & 2m_x \leq n \leq 1 \\ 1 & n > 1 \end{cases} \qquad (188)
\end{aligned}
$$

The above is the distribution function of a Uniform$(2m_x, 1)$ random variable, so we have shown that when $m_x < 0.5$,

$$
p_x^{(2)} \mid f_x = 2k_x - 1 \;\sim\; \text{Uniform}(2m_x, 1), \qquad (189)
$$

i.e., the conditional distribution of $p_x^{(2)}$ given $f_x = 2k_x - 1$ is exactly the same as that of $p_x^{(1)}$ given $f_x = 2k_x - 1$.

Repeating the same argument for the case $m_x < 0.5$ and $f_x = 2k_x$, we have from (185) and (172)

$$
\begin{aligned}
& P(p_x^{(2)} \leq n \mid f_x = 2k_x) = 1 - P(p_x^{(1)} \leq x/v_1 - 2k_x - n \mid f_x = 2k_x) \\
= \; & 1 - \begin{cases} 0 & x/v_1 - 2k_x - n < 0 \\ \frac{x/v_1 - 2k_x - n}{2m_x} & 0 \leq x/v_1 - 2k_x - n \leq 2m_x \\ 1 & x/v_1 - 2k_x - n > 2m_x \end{cases} \\
= \; & 1 - \begin{cases} 0 & n > x/v_1 - 2k_x \\ \frac{x/v_1 - 2k_x - n}{2m_x} & x/v_1 - (2k_x + 2m_x) \leq n \leq x/v_1 - 2k_x \\ 1 & n < x/v_1 - (2k_x + 2m_x) \end{cases} \\
= \; & 1 - \begin{cases} 0 & n > 2m_x \\ \frac{2m_x - n}{2m_x} & 0 \leq n \leq 2m_x \\ 1 & n < 0 \end{cases} \quad \begin{matrix} \text{(due} \\ \text{to} \\ \text{(187)} \end{matrix} \\
= \; & \begin{cases} 0 & n < 0 \\ \frac{n}{2m_x} & 0 \leq n \leq 2m_x \\ 1 & n > 2m_x \end{cases} \qquad (190)
\end{aligned}
$$

The above is the expression for the distribution function of a Uniform$(0, 2m_x)$ random variable, so we have shown that when $m_x < 0.5$, the conditional distribution of $p_x^{(2)}$ given

$f_x = 2k_x$ is given by

$$p_x^{(2)} |\ f_x = 2k_x \ \sim\ \text{Uniform}(0, 2m_x), \tag{191}$$

i.e., the same as that of $p_x^{(1)}$ given $f_x = 2k_x$.

When $m_x \geq 0.5$ and $f_x = 2k_x + 1$, from (185) and (176), we have

$$\begin{aligned}
& P(p_x^{(2)} \leq n \mid f_x = 2k_x + 1) = 1 - P(p_x^{(1)} \leq x/v_1 - 2k_x - 1 - n \mid f_x = 2k_x + 1) \\
= \ & 1 - \begin{cases} 0 & x/v_1 - 2k_x - 1 - n < 0 \\ \frac{x/v_1 - 2k_x - 1 - n}{2m_x - 1} & 0 \leq x/v_1 - 2k_x - 1 - n \leq 2m_x - 1 \\ 1 & x/v_1 - 2k_x - 1 - n > 2m_x - 1 \end{cases} \\
= \ & 1 - \begin{cases} 0 & n > x/v_1 - 2k_x - 1 \\ \frac{x/v_1 - 2k_x - 1 - n}{2m_x - 1} & x/v_1 - (2k_x + 2m_x) \leq n \leq x/v_1 - 2k_x - 1 \\ 1 & n < x/v_1 - (2k_x + 2m_x) \end{cases} \\
= \ & 1 - \begin{cases} 0 & n > 2m_x - 1 & \text{(due} \\ \frac{2m_x - 1 - n}{2m_x - 1} & 0 \leq n \leq 2m_x - 1 & \text{to} \\ 1 & n < 0 & \text{(187))} \end{cases} \\
= \ & \begin{cases} 0 & n < 0 \\ \frac{n}{2m_x - 1} & 0 \leq n \leq 2m_x - 1 \\ 1 & n > 2m_x - 1 \end{cases}
\end{aligned} \tag{192}$$

The above expression is the distribution function of a Uniform$(0, 2m_x - 1)$ random variable, so we have shown that when $m_x \geq 0.5$, the conditional distribution of $p_x^{(2)}$ given $f_x = 2k_x + 1$ is given by

$$p_x^{(2)} |\ f_x = 2k_x + 1 \ \sim\ \text{Uniform}(0, 2m_x - 1), \tag{193}$$

i.e., the same as that of $p_x^{(1)}$ given $f_x = 2k_x + 1$.

Finally, when $m_x \geq 0.5$ and $f_x = 2k_x$, from (185) and (182), we have

$$\begin{aligned}
& P(p_x^{(2)} \leq n \mid f_x = 2k_x) = 1 - P(p_x^{(1)} \leq x/v_1 - 2k_x - n \mid f_x = 2k_x) \\
= \ & 1 - \begin{cases} 0 & x/v_1 - 2k_x - n < 2m_x - 1 \\ \frac{x/v_1 - 2k_x - n - (2m_x - 1)}{2(1 - m_x)} & 2m_x - 1 \leq x/v_1 - 2k_x - n \leq 1 \\ 1 & x/v_1 - 2k_x - n > 1 \end{cases} \\
= \ & 1 - \begin{cases} 0 & n > x/v_1 - (2k_x + 2m_x) + 1 \\ \frac{x/v_1 - (2k_x + 2m_x) - n + 1}{2(1 - m_x)} & x/v_1 - 2k_x - 1 \leq n \leq x/v_1 - (2k_x + 2m_x) + 1 \\ 1 & n < x/v_1 - 2k_x - 1 \end{cases} \\
= \ & 1 - \begin{cases} 0 & n > 1 & \text{(due} \\ \frac{1 - n}{2(1 - m_x)} & 2m_x - 1 \leq n \leq 1 & \text{to} \\ 1 & n < 2m_x - 1 & \text{(187)} \end{cases} \\
= \ & \begin{cases} 0 & n < 2m_x - 1 \\ \frac{n - (2m_x - 1)}{2(1 - m_x)} & 2m_x - 1 \leq n \leq 1 \\ 1 & n > 1 \end{cases}
\end{aligned} \tag{194}$$

The above expression is the distribution function of a Uniform$(2m_x - 1, 1)$ random variable, so we have shown that when $m_x \geq 0.5$, the conditional distribution of $p_x^{(2)}$ given $f_x = 2k_x$ is given by

$$p_x^{(2)} |\ f_x = 2k_x \ \sim\ \text{Uniform}(2m_x - 1, 1), \tag{195}$$

i.e., the same as that of $p_x^{(1)}$ given $f_x = 2k_x$.

The conditional distributions of $p_y^{(1)}$ and $p_y^{(2)}$ given f_y will be exactly analogous to those of $p_x^{(1)}$ and $p_x^{(2)}$ given f_x just derived, with the x quantities replaced by the y analogs.

Thus, in summary, we have the following results:

When $m_x < 0.5$,

$$p_x^{(i)} \mid f_x = 2k_x - 1 \sim U(2m_x, 1), \quad i = 1, 2 \tag{196}$$

and

$$p_x^{(i)} \mid f_x = 2k_x \sim U(0, 2m_x), \quad i = 1, 2. \tag{197}$$

When $m_x \geq 0.5$,

$$p_x^{(i)} \mid f_x = 2k_x + 1 \sim U(0, 2m_x - 1), \quad i = 1, 2 \tag{198}$$

and

$$p_x^{(i)} \mid f_x = 2k_x \sim U(2m_x - 1, 1), \quad i = 1, 2. \tag{199}$$

When $m_y < 0.5$,

$$p_y^{(i)} \mid f_y = 2k_y - 1 \sim U(2m_y, 1), \quad i = 1, 2 \tag{200}$$

and

$$p_y^{(i)} \mid f_x = 2k_y \sim U(0, 2m_y), \quad i = 1, 2. \tag{201}$$

When $m_y \geq 0.5$,

$$p_y^{(i)} \mid f_y = 2k_y + 1 \sim U(0, 2m_y - 1), \quad i = 1, 2 \tag{202}$$

and

$$p_y^{(i)} \mid f_y = 2k_y \sim U(2m_y - 1, 1), \quad i = 1, 2. \tag{203}$$

Appendix C Generating From Distributions of f_x, f_y and f_z

C.1 Generating from an Arbitrary Discrete Distribution

In general, a sample from a discrete distribution of a random variable X given by

$$X = \begin{cases} x_1 & \text{with probability } p_1 \\ x_2 & \text{with probability } p_2 \\ \dots & \dots \\ x_n & \text{with probability } p_n \end{cases} \tag{204}$$

can be done as follows:

1. Generate $U \sim$ Uniform(0,1)
2. Find $i = 1, ..., n$ such that
$$\sum_{j=0}^{i-1} p_j \leq U < \sum_{j=0}^{i} p_j, \tag{205}$$
with $p_0 \equiv 0$.
3. Take the sample equal to x_i.

To generate a string of values from this distribution, one can repeat steps 1–3 a number of times.

This works because by construction, the event $X = x_i$ is equivalent to the event $\sum_{j=0}^{i-1} p_j \leq U < \sum_{j=0}^{i} p_j$, and since $U \sim$ Uniform(0,1),

$$P(X = x_i) = P\left(\sum_{j=0}^{i-1} p_j \leq U < \sum_{j=0}^{i} p_j\right) = \sum_{j=1}^{i} p_j - \sum_{j=0}^{i-1} p_j = p_i, \tag{206}$$

as required.

C.2 Generating From Distributions of f_x, f_y and f_z

As shown in Section 2.1, the distribution of f_x is given as follows:

$$f_x = \begin{cases} f_x^{(1)} & \text{with probability } p(f_x^{(1)}) \\ f_x^{(2)} & \text{with probability } p(f_x^{(2)}) \end{cases} \tag{207}$$

where

$$\begin{aligned} f_x^{(1)} &= 2k_x, \\ f_x^{(2)} &= \begin{cases} 2k_x - 1 & \text{if } m_x < 0.5 \\ 2k_x + 1 & \text{if } m_x \geq 0.5 \end{cases} \end{aligned} \tag{208}$$

$$
\begin{aligned}
p(f_x^{(1)}) &= \begin{cases} 2m_x & \text{if } m_x < 0.5 \\ 2(1-m_x) & \text{if } m_x \geq 0.5 \end{cases} \\
p(f_x^{(2)}) &= 1 - p(f_x^{(1)}). \qquad (209)
\end{aligned}
$$

Thus, following Appendix C.1, a sample from the distribution of f_x can be generated as follows:

1. Generate $U \sim$ Uniform(0,1)
2. If $U < p(f_x^{(1)})$, take the sample equal to $f_x^{(1)}$. Otherwise, take the sample equal to $f_x^{(2)}$.

Sampling from the distributions of f_y and f_z can be done in an equivalent way with x quantities replaced by the y and z quantities, respectively.

Appendix D Object Dimension and Voxel Dimension

This section gives detailed arguments and calculations relevant to the situation when the object dimension is smaller than the corresponding voxel dimension (in one-dimensional setting this translates to $x < v_1$). In addition, this section shows proof of the claim that f_x, as defined in (1), is guaranteed to be non-negative when $x \geq v_1$.

D.1 Non-negativity of f_x when $x \geq v_1$

We show that as long as $x \geq v_1$, i.e., as long as the object size in is no smaller than the voxel size in one-dimensional setting, the expression for f_x in (1) is guaranteed to be non-negative. The number of fully covered voxels f_x was defined there as

$$f_x = \left(\left\lfloor \frac{c_x + x/2}{v_1} \right\rfloor - \left\lceil \frac{c_x - x/2}{v_1} \right\rceil \right), \tag{210}$$

and was shown in Section 2.1 to take on the following 3 possible values:

$$f_x = \begin{cases} 2k_x & \text{if } q_x < min(1 - m_x, m_x) \text{ or } q_x \geq max(1 - m_x, m_x) \\ 2k_x - 1 & \text{if } m_x \leq q_x < 1 - m_x \\ 2k_x + 1 & \text{if } 1 - m_x \leq q_x < m_x \end{cases} \tag{211}$$

Since $k_x \geq 0$, the only one of these values that can be negative is $2k_x - 1$ in case $k_x = 0$. This happens if $m_x \leq q_x < 1 - m_x$. Now, $m_x < 1 - m_x$ if and only if $m_x < 1/2$, or equivalently, if and only if $\frac{x/2}{v_1} < 1/2$ since $k_x = 0$ implies that $m_x = \frac{x/2}{v_1}$. That in turn is true if and only if $x < v_1$. Therefore, as long as $x \geq v_1$, the expression in (1) is guaranteed to be non-negative for $x \geq v_1$.

D.2 Number of Partially Filled Voxels N_p in 2-D

D.2.1 1 small dimension

In this case, one of the following is true:

$x \geq v_1$ and $y < v_2$

$x < v_1$ and $y \geq v_2$

Since the two dimensions are interchangeable, we will denote the small dimension with y and the other dimension with x. Thus, $x \geq v_1$ and $y < v_2$. In this case, there are two subcases: 1 or 2 partially filled voxel lengths along the y dimension. As discussed in Section 2.1.3, the former subcase is equivalent to the event that $m_y \leq q_y \leq 1 - m_y$, while the latter is equivalent to its complement, i.e., the event that $0 \leq q_y < m_y$ or $1 - m_y < q_y \leq 1$. By referring to Figure 4, it is easily seen that

$$N_p = \begin{cases} f_x + 2 & \text{if } m_y \leq q_y \leq 1 - m_y \\ 2(f_x + 2) & \text{otherwise} \end{cases} \tag{212}$$

with $q_y = c_y/v_2$, as before. As discussed at the end of Section 2.1.3, the two events occur with probability $1 - 2m_y$ and $2m_y$, respectively. Moreover, when $y < v_2$, $m_y = \frac{y/2}{v_2} < 0.5$, so $m_y = frac\left(\frac{y/2}{v_2}\right) = \frac{y/2}{v_2}$, leading to $2m_y = y/v_2$. Therefore, conditional on the value of f_x, the distribution of N_p is given by

$$N_p = \begin{cases} f_x + 2 & \text{with probability } 1 - y/v_2 \\ 2(f_x + 2) & \text{with probability } y/v_2 \end{cases} \tag{213}$$

This implies that the (unconditional) distribution of N_p is given by

$$N_p = \begin{cases} f_x^{(1)} + 2 & \text{with probability } p(f_x^{(1)}) \cdot (1 - y/v_2) \\ f_x^{(2)} + 2 & \text{with probability } p(f_x^{(2)}) \cdot (1 - y/v_2) \\ 2(f_x^{(1)} + 2) & \text{with probability } p(f_x^{(1)}) \cdot (y/v_2) \\ 2(f_x^{(2)} + 2) & \text{with probability } p(f_x^{(2)}) \cdot (y/v_2) \end{cases} \tag{214}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22), and $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23).

Although the expected value of N_p can be obtained directly from (214), it is easier to obtain it from (212) using the formula for the iterated expectation (for any two random variables A and B, $E(A) = E_B(E(A|B)) = \sum_b E(A|B = b)P(B = b)$; see [3] for more detail):

$$\begin{aligned} E(N_p) &= E(N_p \text{ given 1 partially filled voxel along } y) \cdot P(1 \text{ partially filled voxel along } y) \\ &+ E(N_p \text{ given 2 partially filled voxels along } y) \cdot P(2 \text{ partially filled voxels along } y) \\ &= E(f_x + 2)\left(1 - \frac{y}{v_2}\right) + E(2(f_x + 2)) \cdot \frac{y}{v_2} = E(f_x + 2)\left(1 - \frac{y}{v_2} + 2 \cdot \frac{y}{v_2}\right) \\ &= (E(f_x) + 2)\left(1 + \frac{y}{v_2}\right) = \left(\frac{x}{v_1} - 1 + 2\right)\left(1 + \frac{y}{v_2}\right) = \\ &= \left(\frac{x}{v_1} + 1\right)\left(\frac{y}{v_2} + 1\right) \end{aligned} \tag{215}$$

(recall that $E(f_x) = x/v_1 - 1$, as shown in Appendix A). Using the iterated expectation again, we can obtain $E(N_p^2)$, as follows:

$$\begin{aligned} E(N_p^2) &= E(N_p^2 \text{ given 1 partially filled voxel along } y) \cdot P(1 \text{ partially filled voxel along } y) \\ &+ E(N_p^2 \text{ given 2 partially filled voxels along } y) \cdot P(2 \text{ partially filled voxels along } y) \\ &= \left(E(f_x + 2)^2\right)\left(1 - \frac{y}{v_2}\right) + \left(E(2(f_x + 2))^2\right) \cdot \frac{y}{v_2} = \left(E(f_x + 2)^2\right)\left(1 - \frac{y}{v_2} + 4 \cdot \frac{y}{v_2}\right) \\ &= \left(E(f_x + 2)^2\right)\left(1 + \frac{3y}{v_2}\right) = \left(Var(f_x + 2) + (E(f_x + 2))^2\right)\left(1 + \frac{3y}{v_2}\right) \\ &= \left(Var(f_x) + (E(f_x) + 2)^2\right)\left(1 + \frac{3y}{v_2}\right) = \left(Var(f_x) + \left(\frac{x}{v_1} + 1\right)^2\right)\left(1 + \frac{3y}{v_2}\right) \end{aligned} \tag{216}$$

Thus, using (215) and (216), the variance of N_p is equal to

$$\begin{aligned}
Var(N_p) &= E(N_p^2) - (E(N_p))^2 \\
&= \left(Var(f_x) + \left(\frac{x}{v_1}+1\right)^2\right)\left(1+\frac{3y}{v_2}\right) - \left(\frac{x}{v_1}+1\right)^2\left(1+\frac{y}{v_2}\right)^2 \\
&= Var(f_x)\left(1+\frac{3y}{v_2}\right) + \left(\frac{x}{v_1}+1\right)^2\left(1+\frac{3y}{v_2} - \left(1+\frac{y}{v_2}\right)^2\right) \\
&= Var(f_x)\left(1+\frac{3y}{v_2}\right) + \left(\frac{x}{v_1}+1\right)^2 \cdot \left(1-\frac{y}{v_2}\right)\cdot\frac{y}{v_2} \\
&= 4\cdot|m_x-0.5|\cdot(0.5-|m_x-0.5|)\cdot\left(1+\frac{3y}{v_2}\right) + \frac{y}{v_2}\left(1-\frac{y}{v_2}\right)\left(\frac{x}{v_1}+1\right)^2
\end{aligned} \tag{217}$$

since $Var(f_x) = 4\cdot|m_x-0.5|\cdot(0.5-|m_x-0.5|)$, as shown in Appendix A). The standard deviation of N_p is then simply the square root of the expression in (217).

D.2.2 2 small dimensions

In this case, we have $x < v_1$ and $y < v_2$, and there are 4 subcases: there can be 1 or 2 partially voxel lengths along each of the two dimensions. Therefore, the number of partially covered voxels is given by

$$N_p = \begin{cases} 1 & \text{if } m_x \le q_x \le 1-m_x \text{ and } m_y \le q_y \le 1-m_y \\ 2 & \text{if } m_x \le q_x \le 1-m_x \text{ and } (0 \le q_y < m_y \text{ or } 1-m_y < q_y \le 1) \\ 2 & \text{if } m_y \le q_y \le 1-m_y \text{ and } (0 \le q_x < m_x \text{ or } 1-m_x < q_x \le 1) \\ 4 & \text{if } (0 \le q_x < m_x \text{ or } 1-m_x < q_x \le 1) \text{ and } (0 \le q_y < m_y \text{ or } 1-m_y < q_y \le 1) \end{cases} \tag{218}$$

where $q_x = c_x/v_1$ and $q_y = c_y/v_2$, as before. Since q_x and q_y are independent and Uniform(0,1), and $x < v_1$ implies that $2m_x = x/v_1$ and $y < v_2$ implies that $2m_y = y/v_2$, the distribution of N_p is given by

$$N_p = \begin{cases} 1 & \text{with probability } \left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right) \\ 2 & \text{with probability } \frac{x}{v_1}\left(1-\frac{y}{v_2}\right) \\ 2 & \text{with probability } \frac{y}{v_2}\left(1-\frac{x}{v_1}\right) \\ 4 & \text{with probability } \frac{x}{v_1}\cdot\frac{y}{v_2} \end{cases} \tag{219}$$

or

$$N_p = \begin{cases} 1 & \text{with probability } \left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right) \\ 2 & \text{with probability } \frac{x}{v_1}\left(1-\frac{y}{v_2}\right) + \frac{y}{v_2}\left(1-\frac{x}{v_1}\right) \\ 4 & \text{with probability } \frac{x}{v_1}\cdot\frac{y}{v_2} \end{cases} \tag{220}$$

Hence, the expected value of the number of partially covered voxels in this case is given by

$$\begin{aligned} E(N_p) &= \left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)+2\cdot\frac{y}{v_2}\left(1-\frac{x}{v_1}\right)+2\cdot\frac{x}{v_1}\left(1-\frac{y}{v_2}\right)+4\cdot\frac{x}{v_1}\frac{y}{v_2} \\ &= \left(\frac{x}{v_1}+1\right)\left(\frac{y}{v_2}+1\right). \end{aligned} \tag{221}$$

Also,

$$\begin{aligned} E(N_p^2) &= \left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)+4\cdot\frac{y}{v_2}\left(1-\frac{x}{v_1}\right)+4\cdot\frac{x}{v_1}\left(1-\frac{y}{v_2}\right)+16\cdot\frac{x}{v_1}\frac{y}{v_2} \\ &= \left(\frac{3x}{v_1}+1\right)\left(\frac{3y}{v_2}+1\right), \end{aligned} \tag{222}$$

so the variance of N_p is given by

$$\begin{aligned} Var(N_p) &= E(N_p^2)-(E(N_p))^2 \\ &= \left(\frac{3x}{v_1}+1\right)\left(\frac{3y}{v_2}+1\right)-\left(\frac{x}{v_1}+1\right)^2\left(\frac{y}{v_2}+1\right)^2. \end{aligned} \tag{223}$$

The standard deviation is the square root of the expression in (223).

D.3 Number of Partially Filled Voxels N_p in 3-D

D.3.1 1 small dimension

Since the three dimensions are interchangeable, we will denote the small dimension with z. Thus, $x \geq v_1$, $y \geq v_2$ and $z < v_3$ In this case, there are two subcases: 1 or 2 partially filled voxel lengths along the z dimension. Following the results obtained in Section 2.1.3, the former subcase is equivalent to the event that $m_z \leq q_z \leq 1-m_z$, while the latter is equivalent to its complement, i.e., the event that $0 \leq q_z < m_z$ or $1-m_z < q_z \leq 1$. Extending the results in Appendix D.2, we have that

$$N_p = \begin{cases} (f_x+2)(f_y+2) & \text{if } m_z \leq q_z \leq 1-m_z \\ 2(f_x+2)(f_y+2) & \text{otherwise} \end{cases} \tag{224}$$

with $q_z = c_z/v_3$, as before. As discussed at the end of Section 2.1.3, the two events occur with probability $1-2m_z$ and $2m_z$, respectively. Moreover, when $z < v_3$, $m_z = \frac{z/2}{v_3} < 0.5$, so $m_z = frac\left(\frac{z/2}{v_3}\right) = \frac{z/2}{v_3}$, leading to $2m_z = z/v_3$. Therefore, conditional on the values of f_x and f_y, the distribution of N_p is given by

$$N_p = \begin{cases} (f_x+2)(f_y+2) & \text{with probability } 1-z/v_3 \\ 2(f_x+2)(f_y+2) & \text{with probability } z/v_3 \end{cases} \tag{225}$$

This implies that the (unconditional) distribution of N_p is given by

$$N_p = \begin{cases} (f_x^{(1)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot (1-z/v_3) \\ (f_x^{(1)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot (1-z/v_3) \\ (f_x^{(2)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot (1-z/v_3) \\ (f_x^{(2)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot (1-z/v_3) \\ 2(f_x^{(1)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(1)}) \cdot (z/v_3) \\ 2(f_x^{(1)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(1)}) \cdot p(f_y^{(2)}) \cdot (z/v_3) \\ 2(f_x^{(2)}+2)(f_y^{(1)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(1)}) \cdot (z/v_3) \\ 2(f_x^{(2)}+2)(f_y^{(2)}+2) & \text{with probability } p(f_x^{(2)}) \cdot p(f_y^{(2)}) \cdot (z/v_3) \end{cases} \tag{226}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22), $f_y^{(1)}$ and $f_y^{(2)}$ as defined in (36), $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23) and $p(f_y^{(1)})$ and $p(f_y^{(2)})$ as defined in (37).

Although the expected value of N_p can be obtained directly from (226), it is easier to obtain it from (224) using the formula for the iterated expectation (for any two random variables A and B, $E(A) = E_B(E(A|B)) = \sum_b E(A|B=b)P(B=b)$; see [3] for more detail):

$$\begin{aligned} E(N_p) &= E(N_p \text{ given 1 partially filled voxel along } z) \cdot P(1 \text{ partially filled voxel along } z) \\ &+ E(N_p \text{ given 2 partially filled voxels along } z) \cdot P(2 \text{ partially filled voxels along } z) \\ &= E[(f_x+2)(f_y+2)]\left(1-\frac{z}{v_3}\right) + E[2(f_x+2)(f_y+2)] \cdot \frac{z}{v_3} \\ &= E[(f_x+2)(f_y+2)]\left(1-\frac{z}{v_3}+2\cdot\frac{z}{v_3}\right) \\ &= E(f_x+2)E(f_y+2)\left(1+\frac{z}{v_3}\right) = \left(\frac{x}{v_1}-1+2\right)\left(\frac{y}{v_2}-1+2\right)\left(1+\frac{z}{v_3}\right) = \\ &= \left(\frac{x}{v_1}+1\right)\left(\frac{y}{v_2}+1\right)\left(\frac{z}{v_3}+1\right) \end{aligned} \tag{227}$$

(Recall that f_x and f_y are independent, so $E[(f_x+2)(f_y+2)] = E(f_x+2)E(f_y+2)$. Moreover, $E(f_x) = x/v_1 - 1$, as shown in Appendix A and analogously, $E(f_y) = y/v_2 - 1$.)

Using the iterated expectation again, we can obtain $E(N_p^2)$, as follows:

$$\begin{aligned}
E(N_p^2) &= E(N_p^2 \text{ given 1 partially filled voxel along } z) \cdot P(1 \text{ partially filled voxel along } z) \\
&+ E(N_p^2 \text{ given 2 partially filled voxels along } z) \cdot P(2 \text{ partially filled voxels along } z) \\
&= \left(E((f_x+2)(f_y+2))^2\right)\left(1-\frac{z}{v_3}\right) + \left(E(2(f_x+2)(f_y+2))^2\right) \cdot \frac{z}{v_3} \\
&= \left(E((f_x+2)(f_y+2))^2\right)\left(1-\frac{z}{v_3}+4\cdot\frac{z}{v_3}\right) \\
&= \left(E(f_x+2)^2\right)\left(E(f_Y+2)^2\right)\left(1+\frac{3z}{v_3}\right) \\
&= \left(Var(f_x+2)+(E(f_x+2))^2\right)\left(Var(f_y+2)+(E(f_y+2))^2\right)\left(1+\frac{3z}{v_3}\right) \\
&= \left(Var(f_x)+(E(f_x)+2)^2\right)\left(Var(f_y)+(E(f_y)+2)^2\right)\left(1+\frac{3z}{v_3}\right) \\
&= \left(Var(f_x)+\left(\frac{x}{v_1}+1\right)^2\right)\left(Var(f_y)+\left(\frac{y}{v_2}+1\right)^2\right)\left(1+\frac{3z}{v_3}\right) \qquad (228)
\end{aligned}$$

Thus, using (227) and (228), the variance of N_p is equal to

$$\begin{aligned}
Var(N_p) &= E(N_p^2) - (E(N_p))^2 \\
&= \left(Var(f_x)+\left(\frac{x}{v_1}+1\right)^2\right)\cdot\left(Var(f_y)+\left(\frac{y}{v_2}+1\right)^2\right)\cdot\left(1+\frac{3z}{v_3}\right) \\
&- \left(\frac{x}{v_1}+1\right)^2\left(1+\frac{y}{v_2}\right)^2\left(1+\frac{z}{v_3}\right)^2 \\
&= \left(4\cdot|m_x-0.5|\cdot(0.5-|m_x-0.5|)+\left(\frac{x}{v_1}+1\right)^2\right) \\
&\times \left(4\cdot|m_y-0.5|\cdot(0.5-|m_y-0.5|)+\left(\frac{y}{v_2}+1\right)^2\right)\cdot\left(1+\frac{3z}{v_3}\right) \\
&- \left(\frac{x}{v_1}+1\right)^2\left(1+\frac{y}{v_2}\right)^2\left(1+\frac{z}{v_3}\right)^2 \qquad (229)
\end{aligned}$$

since $Var(f_x) = 4 \cdot |m_x - 0.5| \cdot (0.5 - |m_x - 0.5|)$, as shown in Appendix A, and $Var(f_y) = 4 \cdot |m_y - 0.5| \cdot (0.5 - |m_y - 0.5|)$, by analogy. The standard deviation of N_p is then simply the square root of the expression in (229).

D.3.2 2 small dimensions

Since the three dimensions are interchangeable, we will denote the two small dimensions with y and z. Thus, $x \geq v_1$, $y < v_2$ and $z < v_3$. In this case, there are 4 subcases: there can

be 1 or 2 partially voxel lengths along each of the two small dimensions. Since $x \geq v_1$, there are $f_x + 2$ voxel lengths covered along the x dimension. Therefore, the number of partially covered voxels is given by

$$N_p = \begin{cases} f_x + 2 & \text{if } m_y \leq q_y \leq 1 - m_y \text{ and } m_z \leq q_z \leq 1 - m_z \\ 2(f_x + 2) & \text{if } m_y \leq q_y \leq 1 - m_y \text{ and } (0 \leq q_z < m_z \text{ or } 1 - m_z < q_z \leq 1) \\ 2(f_x + 2) & \text{if } m_z \leq q_z \leq 1 - m_z \text{ and } (0 \leq q_y < m_y \text{ or } 1 - m_y < q_y \leq 1) \\ 4(f_x + 2) & \text{if } (0 \leq q_y < m_y \text{ or } 1 - m_y < q_y \leq 1) \text{ and } (0 \leq q_z < m_z \text{ or } 1 - m_z < q_z \leq 1) \end{cases} \tag{230}$$

where $q_y = c_y/v_2$ and $q_z = c_z/v_3$, as before. Since q_y and q_z are independent and Uniform(0,1), and $y < v_2$ implies that $2m_y = y/v_2$ and $z < v_3$ implies that $2m_z = z/v_3$, conditional on f_x, the distribution of N_p is given by

$$N_p = \begin{cases} f_x + 2 & \text{with probability } \left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 2(f_x + 2) & \text{with probability } \frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) \\ 2(f_x + 2) & \text{with probability } \frac{z}{v_3}\left(1 - \frac{y}{v_2}\right) \\ 4(f_x + 2) & \text{with probability } \frac{y}{v_2} \cdot \frac{z}{v_3} \end{cases} \tag{231}$$

or

$$N_p = \begin{cases} f_x + 2 & \text{with probability } \left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 2(f_x + 2) & \text{with probability } \frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) + \frac{z}{v_3}\left(1 - \frac{y}{v_2}\right) \\ 4(f_x + 2) & \text{with probability } \frac{y}{v_2} \cdot \frac{z}{v_3} \end{cases} \tag{232}$$

Hence, the (unconditional) distribution of N_p is given by

$$N_p = \begin{cases} f_x^{(1)} + 2 & \text{with probability } p(f_x^{(1)}) \cdot \left(1 - \frac{y}{v_2}\right) \cdot \left(1 - \frac{z}{v_3}\right) \\ f_x^{(2)} + 2 & \text{with probability } p(f_x^{(2)}) \cdot \left(1 - \frac{y}{v_2}\right) \cdot \left(1 - \frac{z}{v_3}\right) \\ 2(f_x^{(1)} + 2) & \text{with probability } p(f_x^{(1)}) \cdot \left(\frac{y}{v_2} \cdot \left(1 - \frac{z}{v_3}\right) + \frac{z}{v_3}\left(1 - \frac{y}{v_2}\right)\right) \\ 2(f_x^{(2)} + 2) & \text{with probability } p(f_x^{(2)}) \cdot \left(\frac{y}{v_2} \cdot \left(1 - \frac{z}{v_3}\right) + \frac{z}{v_3}\left(1 - \frac{y}{v_2}\right)\right) \\ 4(f_x^{(1)} + 2) & \text{with probability } p(f_x^{(1)}) \cdot \frac{y}{v_2} \cdot \frac{z}{v_3} \\ 4(f_x^{(2)} + 2) & \text{with probability } p(f_x^{(2)}) \cdot \frac{y}{v_2} \cdot \frac{z}{v_3} \end{cases} \tag{233}$$

with $f_x^{(1)}$ and $f_x^{(2)}$ as defined in (22) and $p(f_x^{(1)})$ and $p(f_x^{(2)})$ as defined in (23).

From (232), using the iterated expectation formula, we have that the expected value of

the number of partially covered voxels in this case is given by

$$\begin{aligned}
E(N_p) &= E(f_x+2)\cdot\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right) \\
&+ E(2(f_x+2))\cdot\left(\frac{y}{v_2}\left(1-\frac{z}{v_3}\right)+\frac{z}{v_3}\left(1-\frac{y}{v_2}\right)\right) \\
&+ E(4(f_x+2))\cdot\frac{y}{v_2}\cdot\frac{z}{v_3} \\
&= E(f_x+2)\left(\frac{y}{v_2}+1\right)\left(\frac{z}{v_3}+1\right) \\
&= \left(\frac{x}{v_1}+1\right)\left(\frac{y}{v_2}+1\right)\left(\frac{z}{v_3}+1\right). \qquad (234)
\end{aligned}$$

Also,

$$\begin{aligned}
E(N_p^2) &= E((f_x+2)^2)\cdot\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right) \\
&+ E(4(f_x+2)^2)\cdot\left(\frac{y}{v_2}\left(1-\frac{z}{v_3}\right)+\frac{z}{v_3}\left(1-\frac{y}{v_2}\right)\right) \\
&+ E(16(f_x+2)^2)\cdot\frac{y}{v_2}\cdot\frac{z}{v_3} \\
&= E((f_x+2)^2)\left(\frac{3y}{v_2}+1\right)\left(\frac{3z}{v_3}+1\right) \\
&= (Var(f_x)+(E(f_x+2))^2)\left(\frac{3y}{v_2}+1\right)\left(\frac{3z}{v_3}+1\right) \\
&= \left(Var(f_x)+\left(\frac{x}{v_1}+1\right)^2\right)\left(\frac{3y}{v_2}+1\right)\left(\frac{3z}{v_3}+1\right), \qquad (235)
\end{aligned}$$

so the variance of N_p is given by

$$\begin{aligned}
Var(N_p) &= E(N_p^2)-(E(N_p))^2 \\
&= \left(Var(f_x)+\left(\frac{x}{v_1}+1\right)^2\right)\left(\frac{3y}{v_2}+1\right)\left(\frac{3z}{v_3}+1\right) \qquad (236) \\
&- \left(\frac{x}{v_1}+1\right)^2\left(\frac{y}{v_2}+1\right)^2\left(\frac{z}{v_3}+1\right)^2 \\
&= \left(4\cdot|m_x-0.5|\cdot(0.5-|m_x-0.5|)+\left(\frac{x}{v_1}+1\right)^2\right)\left(\frac{3y}{v_2}+1\right)\left(\frac{3z}{v_3}+1\right) \\
&- \left(\frac{x}{v_1}+1\right)^2\left(\frac{y}{v_2}+1\right)^2\left(\frac{z}{v_3}+1\right)^2 \qquad (237)
\end{aligned}$$

The standard deviation is the square root of the expression in (236).

D.3.3 3 small dimensions

Here, we consider the case when $x < v_1$, $y < v_2$ and $z < v_3$. There can be 1 or 2 partially voxel lengths covered along each of the three small dimensions. Thus, there can be 1, 2, 4, and 8 partially covered voxels. The number of partially covered voxels is given by

$$N_p = \begin{cases} 1 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } m_y \le q_y \le 1 - m_y \text{ and } m_z \le q_z \le 1 - m_z \\ 2 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } m_y \le q_y \le 1 - m_y \text{ and } (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \\ 2 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and } m_z \le q_z \le 1 - m_z \\ 2 & \text{if } (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } m_y \le q_y \le 1 - m_y \text{ and } m_z \le q_z \le 1 - m_z \\ 4 & \text{if } m_x \le q_x \le 1 - m_x \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and} \\ & (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \\ 4 & \text{if } (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } m_y \le q_y \le 1 - m_y \text{ and} \\ & (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \\ 4 & \text{if } (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and} \\ & m_z \le q_z \le 1 - m_z \\ 8 & (0 \le q_x < m_x \text{ or } 1 - m_x < q_x \le 1) \text{ and } (0 \le q_y < m_y \text{ or } 1 - m_y < q_y \le 1) \text{ and} \\ & (0 \le q_z < m_z \text{ or } 1 - m_z < q_z \le 1) \end{cases} \tag{238}$$

where $q_x = c_x/v_1$, $q_y = c_y/v_2$ and $q_z = c_z/v_3$, as before. Since q_x, q_y and q_z are independent and Uniform(0,1), and $x < v_1$ implies that $2m_x = x/v_1$, $y < v_2$ implies that $2m_y = y/v_2$, and $z < v_3$ implies that $2m_z = z/v_3$, the distribution of the number of partially covered voxels is given by

$$N_p = \begin{cases} 1 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 2 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\left(1 - \frac{y}{v_2}\right)\frac{z}{v_3} \\ 2 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) \\ 2 & \text{with probability } \frac{x}{v_1}\left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 4 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\frac{y}{v_2}\frac{z}{v_3} \\ 4 & \text{with probability } \frac{x}{v_1}\left(1 - \frac{y}{v_2}\right)\frac{z}{v_3} \\ 4 & \text{with probability } \frac{x}{v_1}\frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) \\ 8 & \text{with probability } \frac{x}{v_1}\frac{y}{v_2}\frac{z}{v_3} \end{cases} \tag{239}$$

or

$$N_p = \begin{cases} 1 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 2 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\left(1 - \frac{y}{v_2}\right)\frac{z}{v_3} + \left(1 - \frac{x}{v_1}\right)\frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) + \frac{x}{v_1}\left(1 - \frac{y}{v_2}\right)\left(1 - \frac{z}{v_3}\right) \\ 4 & \text{with probability } \left(1 - \frac{x}{v_1}\right)\frac{y}{v_2}\frac{z}{v_3} + \frac{x}{v_1}\left(1 - \frac{y}{v_2}\right)\frac{z}{v_3} + \frac{x}{v_1}\frac{y}{v_2}\left(1 - \frac{z}{v_3}\right) \\ 8 & \text{with probability } \frac{x}{v_1}\frac{y}{v_2}\frac{z}{v_3}. \end{cases} \tag{240}$$

The expected value of N_p is given by

$$\begin{aligned}
E(N_p) &= \left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right) \\
&+ 2\cdot\left(\left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\frac{z}{v_3}+\left(1-\frac{x}{v_1}\right)\frac{y}{v_2}\left(1-\frac{z}{v_3}\right)+\frac{x}{v_1}\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right)\right) \\
&+ 4\cdot\left(\left(1-\frac{x}{v_1}\right)\frac{y}{v_2}\frac{z}{v_3}+\frac{x}{v_1}\left(1-\frac{y}{v_2}\right)\frac{z}{v_3}+\frac{x}{v_1}\frac{y}{v_2}\left(1-\frac{z}{v_3}\right)\right) \\
&+ 8\cdot\frac{x}{v_1}\frac{y}{v_2}\frac{z}{v_3} \\
&= \left(\frac{x}{v_1}+1\right)\left(\frac{y}{v_2}+1\right)\left(\frac{z}{v_3}+1\right). \qquad (241)
\end{aligned}$$

Also,

$$\begin{aligned}
E(N_p^2) &= \left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right) \\
&+ 4\cdot\left(\left(1-\frac{x}{v_1}\right)\left(1-\frac{y}{v_2}\right)\frac{z}{v_3}+\left(1-\frac{x}{v_1}\right)\frac{y}{v_2}\left(1-\frac{z}{v_3}\right)+\frac{x}{v_1}\left(1-\frac{y}{v_2}\right)\left(1-\frac{z}{v_3}\right)\right) \\
&+ 16\cdot\left(\left(1-\frac{x}{v_1}\right)\frac{y}{v_2}\frac{z}{v_3}+\frac{x}{v_1}\left(1-\frac{y}{v_2}\right)\frac{z}{v_3}+\frac{x}{v_1}\frac{y}{v_2}\left(1-\frac{z}{v_3}\right)\right) \\
&+ 64\cdot\frac{x}{v_1}\frac{y}{v_2}\frac{z}{v_3} \\
&= \left(\frac{3x}{v_1}+1\right)\left(\frac{3y}{v_2}+1\right)\left(\frac{3z}{v_3}+1\right), \qquad (242)
\end{aligned}$$

so the variance of N_p is given by

$$\begin{aligned}
Var(N_p) &= E(N_p^2)-(E(N_p))^2 \\
&= \left(\frac{3x}{v_1}+1\right)\left(\frac{3y}{v_2}+1\right)\left(\frac{3z}{v_3}+1\right)-\left(\frac{x}{v_1}+1\right)^2\left(\frac{y}{v_2}+1\right)^2\left(\frac{z}{v_3}+1\right)^2. \qquad (243)
\end{aligned}$$

The standard deviation is the square root of the expression in (243).

Appendix E Distributions of $X - \lfloor X \rfloor$ and $\lceil X \rceil - X$ for a Continuous Random Variable X

E.1 General Case

We will first obtain the general formulas for the distributions of the random variables $Y = X - \lfloor X \rfloor$ and $Z = \lceil X \rceil - X$, where X is an arbitrary continuous random variable, and $\lceil X \rceil$ and $\lfloor X \rfloor$ are the ceiling and the floor of X, respectively. Clearly, both Y and Z can only take on values between 0 and 1.

Let C denote the set of all possible values of $\lfloor X \rfloor$. Then for $0 \leq k \leq 1$, the distribution function (df) of Y, denoted by F_Y, is given by

$$\begin{aligned} F_Y(k) &= P(Y \leq k) = P(X - \lfloor X \rfloor \leq k) = \sum_{c \in C} P(X - \lfloor X \rfloor \leq k \ \cap \ \lfloor X \rfloor = c) \\ &= \sum_{c \in C} P(X \leq k + c \ \cap \ \lfloor X \rfloor = c) = \sum_{c \in C} P(X \leq k + c \ \cap \ c \leq X < c + 1) \\ &= \sum_{c \in C} P(c \leq X \leq k + c) = \sum_{c \in C} \left(F_X(k + c) - F_X(c) \right), \end{aligned} \tag{244}$$

where F_X denotes the df of X. Thus, for $0 \leq k \leq 1$, the probability density function (pdf) of $Y = X - \lfloor X \rfloor$, denoted by f_Y, is equal to

$$f_Y(k) = \frac{d}{dk} F_Y(k) = \sum_{c \in C} f_X(k + c), \tag{245}$$

where f_X denotes the pdf of X (i.e., $f_X(k) = \frac{d}{dk} F_X(k)$). For other values of k, the value of the pdf of Y is 0.

Now, let D denote the set of all possible values of $\lceil X \rceil$. Then for $0 \leq k \leq 1$, the distribution function of Z, denoted by F_Z, is given by

$$\begin{aligned} F_Z(k) &= P(Z \leq k) = P(\lceil X \rceil - X \leq k) = \sum_{d \in D} P(\lceil X \rceil - X \leq k \ \cap \ \lceil X \rceil = d) \\ &= \sum_{d \in D} P(X \geq d - k \ \cap \ \lceil X \rceil = d) = \sum_{d \in D} P(X \geq d - k \ \cap \ d - 1 < X \leq d) \\ &= \sum_{d \in D} P(d - k \leq X \leq d) = \sum_{d \in D} \left(F_X(d) - F_X(d - k) \right). \end{aligned} \tag{246}$$

Thus, for $0 \leq k \leq 1$, the pdf of $Z = \lceil X \rceil - X$, denoted by f_Z, is equal to

$$f_Z(k) = \frac{d}{dk} F_Z(k) = \sum_{d \in D} f_X(d - k), \tag{247}$$

where, as before, f_X denotes the pdf of X (for other values of k, the pdf of Z is 0).

E.2 Uniform Case

E.2.1 $Y = X - \lfloor X \rfloor$

Suppose $X \sim \text{Uniform}(a, b)$ with a and b any real numbers so that $a < b$. That is,

$$f_X(x) = \frac{1}{b-a} \quad \text{for} \quad a \leq x \leq b \tag{248}$$

and 0 otherwise. The set C, as defined in Appendix E.1, is given by $C = (\lfloor a \rfloor, \lfloor a \rfloor + 1, ..., \lfloor b \rfloor - 1, \lfloor b \rfloor)$. From (245), we have

$$f_Y(k) = \sum_{c=\lfloor a \rfloor}^{\lfloor b \rfloor} f_X(k+c). \tag{249}$$

According to (248),

$$f_X(k+c) = \frac{1}{b-a} \quad \text{for} \quad a \leq k+c \leq b \tag{250}$$

and 0 otherwise, or

$$f_X(k+c) = \frac{1}{b-a} \quad \text{for} \quad a-k \leq c \leq b-k \tag{251}$$

and 0 otherwise.
Now, $\lfloor a \rfloor + 1 \geq a - k$ since $1 + k \geq a - \lfloor a \rfloor$ as $1 + k \geq 1 \geq a - \lfloor a \rfloor$. So, for any $c \geq \lfloor a \rfloor + 1$, we always have that $c \geq a - k$. Also, $\lfloor b \rfloor - 1 \leq b - k$ since $k - 1 \leq b - \lfloor b \rfloor$ as $k - 1 \leq 0 \leq b - \lfloor b \rfloor$. So, for any $c \leq \lfloor b \rfloor - 1$, we always have that $c \leq b - k$. Therefore, for $c = \lfloor a \rfloor + 1, ..., \lfloor b \rfloor - 1$,

$$f_X(k+c) = \frac{1}{b-a}. \tag{252}$$

Thus, from (249),

$$\begin{aligned} f_Y(k) &= f_X(\lfloor a \rfloor + k) + \sum_{c=\lfloor a \rfloor + 1}^{\lfloor b \rfloor - 1} \left(\frac{1}{b-a} \right) + f_X(\lfloor b \rfloor + k) \\ &= f_X(\lfloor a \rfloor + k) + f_X(\lfloor b \rfloor + k) + \frac{1}{b-a} (\lfloor b \rfloor - 1 - \lfloor a \rfloor) \end{aligned} \tag{253}$$

Now,

$$f_X(\lfloor a \rfloor + k) = \begin{cases} 0 & \lfloor a \rfloor + k < a \quad \Leftrightarrow k < a - \lfloor a \rfloor \\ \frac{1}{b-a} & \lfloor a \rfloor + k \geq a \quad \Leftrightarrow k \geq a - \lfloor a \rfloor \end{cases} \tag{254}$$

and

$$f_X(\lfloor b \rfloor + k) = \begin{cases} 0 & \lfloor b \rfloor + k > b \quad \Leftrightarrow k > b - \lfloor b \rfloor \\ \frac{1}{b-a} & \lfloor b \rfloor + k \leq b \quad \Leftrightarrow k \leq b - \lfloor b \rfloor \end{cases} \tag{255}$$

Therefore, from (253), (254) and (255), we have the following 4 cases:

$$\text{if} \quad k < a - \lfloor a \rfloor \quad \text{and} \quad k < b - \lfloor b \rfloor,$$

$$
\begin{aligned}
f_Y(k) &= 0 + \frac{1}{b-a} + \frac{1}{b-a}\left(\lfloor b \rfloor - 1 - \lfloor a \rfloor\right) \\
&= \frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor\right) \qquad (256)
\end{aligned}
$$

$$\text{if} \;\; k < a - \lfloor a \rfloor \;\; \text{and} \;\; k > b - \lfloor b \rfloor,$$

$$
\begin{aligned}
f_{X-\lfloor X \rfloor}(k) &= 0 + 0 + \frac{1}{b-a}\left(\lfloor b \rfloor - 1 - \lfloor a \rfloor\right) \\
&= \frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor - 1\right) \qquad (257)
\end{aligned}
$$

$$\text{if} \;\; k \geq a - \lfloor a \rfloor \;\; \text{and} \;\; k < b - \lfloor b \rfloor,$$

$$
\begin{aligned}
f_Y(k) &= \frac{1}{b-a} + \frac{1}{b-a} + \frac{1}{b-a}\left(\lfloor b \rfloor - 1 - \lfloor a \rfloor\right) \\
&= \frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor + 1\right) \qquad (258)
\end{aligned}
$$

$$\text{if} \;\; k \geq a - \lfloor a \rfloor \;\; \text{and} \;\; k \geq b - \lfloor b \rfloor,$$

$$
\begin{aligned}
f_Y(k) &= \frac{1}{b-a} + 0 + \frac{1}{b-a} + \frac{1}{b-a}\left(\lfloor b \rfloor - 1 - \lfloor a \rfloor\right) \\
&= \frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rceil\right) \qquad (259)
\end{aligned}
$$

Hence, putting the 4 cases together, we have

$$
f_Y(k) = \begin{cases}
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor\right) & 0 \leq k \leq min\left(a - \lfloor a \rfloor, b - \lfloor b \rfloor\right) \\
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor + 1\right) & a - \lfloor a \rfloor \leq k \leq b - \lfloor b \rfloor \\
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor - 1\right) & b - \lfloor b \rfloor \leq k \leq a - \lfloor a \rfloor \\
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor\right) & max\left(a - \lfloor a \rfloor, b - \lfloor b \rfloor\right) \leq k \leq 1
\end{cases} \qquad (260)
$$

Thus, if $a - \lfloor a \rfloor \leq b - \lfloor b \rfloor$,

$$
f_Y(k) = \begin{cases}
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor\right) & 0 \leq k \leq a - \lfloor a \rfloor \\
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor + 1\right) & a - \lfloor a \rfloor \leq k \leq b - \lfloor b \rfloor \\
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor\right) & b - \lfloor b \rfloor \leq k \leq 1
\end{cases} \qquad (261)
$$

while if $b - \lfloor b \rfloor \leq a - \lfloor a \rfloor$,

$$
f_Y(k) = \begin{cases}
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor\right) & 0 \leq k \leq b - \lfloor b \rfloor \\
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor - 1\right) & b - \lfloor b \rfloor \leq k \leq a - \lfloor a \rfloor \\
\frac{1}{b-a}\left(\lfloor b \rfloor - \lfloor a \rfloor\right) & a - \lfloor a \rfloor \leq k \leq 1
\end{cases} \qquad (262)
$$

E.2.2 $Z = \lceil X \rceil - X$

Suppose $X \sim \text{Uniform}(a, b)$ with a and b any real numbers so that $a < b$. The set D, as defined in Appendix E.1, is given by $D = (\lceil a \rceil, \lceil a \rceil + 1, ..., \lceil b \rceil - 1, \lceil b \rceil)$. From (247), we have

$$f_Z(k) = \sum_{d=\lceil a \rceil}^{\lceil b \rceil} f_X(d-k). \tag{263}$$

According to (248),

$$f_X(d-k) = \frac{1}{b-a} \quad \text{for} \quad a \leq d-k \leq b \tag{264}$$

and 0 otherwise, or

$$f_X(d-k) = \frac{1}{b-a} \quad \text{for} \quad a+k \leq d \leq b+k \tag{265}$$

and 0 otherwise.
Now, $\lceil a \rceil + 1 \geq a + k$ since $1 - k \geq a - \lceil a \rceil$ as $1 - k \geq 0 \geq a - \lceil a \rceil$. So, for any $d \geq \lceil a \rceil + 1$, we always have that $d \geq a + k$. Also, $\lceil b \rceil - 1 \leq b + k$ since $k + 1 \geq \lceil b \rceil - b$ as $k + 1 \geq 1 \geq \lceil b \rceil - b$. So, for any $d \leq \lfloor b \rfloor - 1$, we always have that $d \leq b + k$. Therefore, for $d = \lceil a \rceil + 1, ..., \lceil b \rceil - 1$,

$$f_X(d-k) = \frac{1}{b-a}. \tag{266}$$

Thus, from (263),

$$\begin{aligned} f_Z(k) &= f_X(\lceil a \rceil - k) + \sum_{d=\lceil a \rceil + 1}^{\lceil b \rceil - 1} \left(\frac{1}{b-a} \right) + f_X(\lceil b \rceil - k) \\ &= f_X(\lceil a \rceil - k) + f_X(\lceil b \rceil - k) + \frac{1}{b-a} (\lceil b \rceil - 1 - \lceil a \rceil) \end{aligned} \tag{267}$$

Now,

$$f_X(\lceil a \rceil - k) = \begin{cases} 0 & \lceil a \rceil - k < a \quad \Leftrightarrow k > \lceil a \rceil - a \\ \frac{1}{b-a} & \lceil a \rceil - k \geq a \quad \Leftrightarrow k \leq \lceil a \rceil - a \end{cases} \tag{268}$$

and

$$f_X(\lceil b \rceil - k) = \begin{cases} 0 & \lceil b \rceil - k > b \quad \Leftrightarrow k < \lceil b \rceil - b \\ \frac{1}{b-a} & \lceil b \rceil - k \leq b \quad \Leftrightarrow k \geq \lceil b \rceil - b \end{cases} \tag{269}$$

Therefore, from (267), (268) and (269), we have the following 4 cases:

$$\text{if} \quad k < \lceil a \rceil - a \quad \text{and} \quad k < \lceil b \rceil - b,$$

$$\begin{aligned} f_Z(k) &= \frac{1}{b-a} + 0 + \frac{1}{b-a} (\lceil b \rceil - 1 - \lceil a \rceil) \\ &= \frac{1}{b-a} (\lceil b \rceil - \lceil a \rceil) \end{aligned} \tag{270}$$

$$\text{if} \;\; k < \lceil a \rceil - a \;\; \text{and} \;\; k > \lceil b \rceil - b,$$

$$\begin{aligned} f_Z(k) &= \frac{1}{b-a} + \frac{1}{b-a} + \frac{1}{b-a}\left(\lceil b \rceil - 1 - \lceil a \rceil\right) \\ &= \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil + 1\right) \end{aligned} \tag{271}$$

$$\text{if} \;\; k \geq \lceil a \rceil - a \;\; \text{and} \;\; k < \lceil b \rceil - b,$$

$$\begin{aligned} f_Z(k) &= 0 + 0 + \frac{1}{b-a}\left(\lceil b \rceil - 1 - \lceil a \rceil\right) \\ &= \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil - 1\right) \end{aligned} \tag{272}$$

$$\text{if} \;\; k \geq \lceil a \rceil - a \;\; \text{and} \;\; k \geq \lceil b \rceil - b,$$

$$\begin{aligned} f_Z(k) &= 0 + \frac{1}{b-a} + \frac{1}{b-a}\left(\lceil b \rceil - 1 - \lceil a \rceil\right) \\ &= \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil\right) \end{aligned} \tag{273}$$

Hence, putting the 4 cases together, we have

$$f_Z(k) = \begin{cases} \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil\right) & 0 \leq k \leq min\left(\lceil a \rceil - a, \lceil b \rceil - b\right) \\ \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil - 1\right) & \lceil a \rceil - a \leq k \leq \lceil b \rceil - b \\ \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil + 1\right) & \lceil b \rceil - b \leq k \leq \lceil a \rceil - a \\ \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil\right) & max\left(\lceil a \rceil - a, \lceil b \rceil - a\right) \leq k \leq 1 \end{cases} \tag{274}$$

Thus, if $\lceil a \rceil - a \leq \lceil b \rceil - b$,

$$f_Z(k) = \begin{cases} \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil\right) & 0 \leq k \leq \lceil a \rceil - a \\ \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil - 1\right) & \lceil a \rceil - a \leq k \leq \lceil b \rceil - b \\ \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil\right) & \lceil b \rceil - b \leq k \leq 1 \end{cases} \tag{275}$$

while if $\lceil b \rceil - b \leq \lceil a \rceil - a$,

$$f_Z(k) = \begin{cases} \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil\right) & 0 \leq k \leq \lceil b \rceil - b \\ \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil + 1\right) & \lceil b \rceil - b \leq k \leq \lceil a \rceil - a \\ \frac{1}{b-a}\left(\lceil b \rceil - \lceil a \rceil\right) & \lceil a \rceil - a \leq k \leq 1 \end{cases} \tag{276}$$

E.2.3 Special Uniform Case

The special case of the result derived in Appendix E.2.1 is when $a - \lfloor a \rfloor = b - \lfloor b \rfloor$, while a special case of the result derived in Appendix E.2.2 is when $\lceil a \rceil - a = \lceil b \rceil - b$. In the former case, we have that $\lfloor b \rfloor - \lfloor a \rfloor = b - a$, while in the latter case, we have that $\lceil b \rceil - \lceil a \rceil = b - a$, in other words, in both cases, the length of the interval over which the random variable X is distributed is an integer.

In the case of the random variable $Y = X - \lfloor X \rfloor$, equations (261) and (262) combine to yield

$$f_Y(k) = 1 \quad 0 \leq k \leq 1, \tag{277}$$

that is, Y is Uniform(0,1). Analogously, in the case of the random variable $Z = \lceil X \rceil - X$, equations (275) and (276) combine to yield

$$f_Z(k) = 1 \quad 0 \leq k \leq 1, \tag{278}$$

so, Z, too, is Uniform(0,1). (Note that the case when a and b are both integers is a special case of the special case, so for $X \sim \text{Uniform}(a, b)$ with a and b both integers, Y and Z are both Uniform(0,1).)

E.3 Distributions of $p_x^{(1)}$ and $p_x^{(2)}$ when $x < v_1$

When $x < v_1$, there are 2 partially filled voxels if and only if $0 \leq q_x \leq \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} \leq q_x \leq 1$ (otherwise, there is only 1 partially filled voxel whose fraction covered is fixed at x/v_1). The fraction covered in the voxel on the left is given by

$$p_x^{(1)} = \left\lceil q_x - \frac{x/2}{v_1} \right\rceil - \left(q_x - \frac{x/2}{v_1} \right) \tag{279}$$

while the fraction covered in the voxel on the right is given by

$$p_x^{(2)} = \left(q_x + \frac{x/2}{v_1} \right) - \left\lfloor q_x + \frac{x/2}{v_1} \right\rfloor \tag{280}$$

for $0 \leq q_x \leq \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} \leq q_x \leq 1$.

For ease of notation, let

$$R = \left(q_x - \frac{x/2}{v_1} \right) \tag{281}$$

and

$$S = \left(q_x + \frac{x/2}{v_1} \right), \tag{282}$$

so $p_x^{(1)}$ and $p_x^{(2)}$ become

$$p_x^{(1)} = \lceil R \rceil - R \equiv Z \tag{283}$$

and

$$p_x^{(2)} = S - \lfloor S \rfloor \equiv Y \tag{284}$$

in keeping with notation introduced in Appendix E.1. Since the distributions of $p_x^{(1)}$ and $p_x^{(2)}$ are only relevant when $0 \leq q_x < \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} < q_x \leq 1$, the goal is to obtain the conditional distributions of $Z = \lceil R \rceil - R$ and $Y = S - \lfloor S \rfloor$ given $0 \leq q_x \leq \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} \leq q_x \leq 1$.

E.3.1 Distribution of $p_x^{(1)}$

First, we will obtain the conditional distributions of R given $0 \leq q_x \leq \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} \leq q_x \leq 1$ and use the results we obtained in Appendix E.1 to derive the conditional distribution of $Z = \lceil R \rceil - R$. Recall that q_x is Uniform(0,1), so the probability of q_x being in any subinterval of (0,1) is just the length of that subinterval. With that in mind, the conditional distribution function of R is given by

$$\begin{aligned}
&P(R \leq k \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1) = \\
&P\left(q_x - \tfrac{x/2}{v_1} \leq k \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right) = \\
&P\left(q_x \leq k + \tfrac{x/2}{v_1} \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right) = \\
&P\left(q_x \leq k + \tfrac{x/2}{v_1} \cap \left(0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right)\right) / P\left(0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right) = \\
&P\left(q_x \leq k + \tfrac{x/2}{v_1} \cap \left(0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right)\right) / \left(\tfrac{x}{v_1}\right).
\end{aligned} \tag{285}$$

The numerator $P\left(q_x \leq k + \frac{x/2}{v_1} \cap \left(0 \leq q_x \leq \frac{x/2}{v_1} \text{ or } 1 - \frac{x/2}{v_1} \leq q_x \leq 1\right)\right)$ evaluates to

$$\begin{cases}
P\left(0 \leq q_x \leq k + \frac{x/2}{v_1}\right) & 0 \leq k + \frac{x/2}{v_1} < \frac{x/2}{v_1} \\
P\left(0 \leq q_x \leq \frac{x/2}{v_1}\right) & \frac{x/2}{v_1} \leq k + \frac{x/2}{v_1} < 1 - \frac{x/2}{v_1} \\
P\left(1 - \frac{x/2}{v_1} \leq q_x \leq k + \frac{x/2}{v_1}\right) & 1 - \frac{x/2}{v_1} \leq k + \frac{x/2}{v_1} \leq 1
\end{cases} \tag{286}$$

Thus, the conditional distribution function of R given $0 \leq q_x \leq \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} \leq q_x \leq 1$ is equal to

$$\begin{aligned}
&P(R \leq k \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1) = \\
&= \begin{cases}
\left(k + \frac{x/2}{v_1}\right) / \left(\frac{x}{v_1}\right) & -\frac{x/2}{v_1} \leq k < 0 \\
1/2 & 0 \leq k < 1 - x/v_1 \\
(k - 1 + x/v_1) / \left(\frac{x}{v_1}\right) & 1 - x/v_1 \leq k \leq 1 - \frac{x/2}{v_1}
\end{cases}
\end{aligned} \tag{287}$$

Taking a derivative of the conditional distribution function with respect to k, we have that the conditional density of R is given by

$$f_R(k) = v_1/x, \quad -\frac{x/2}{v_1} \leq k < 0 \text{ or } 1 - \frac{x}{v_1} \leq k \leq 1 - \frac{x/2}{v_1} \tag{288}$$

and 0 otherwise.

Now, using the notation introduced in Appendix E.1, according to (247), we have that the density of $p_x^{(1)} = Z = \lceil R \rceil - R$ is equal to

$$f_Z(k) = f_R(-k) + f_R(1-k) \tag{289}$$

for $0 \leq k \leq x/v_1$ (recall that the upper bound on Z is now x/v_1 rather than 1, as the entire area covered by the two partially covered voxels is equal to x/v_1, so the area covered by either of the two partially covered voxels cannot exceed this value).

Let A and B denote the sets of values k for which $f_R(-k)$ and $f_R(1-k)$, respectively, are nonzero, and let $\overline{A}$ and $\overline{B}$ denote the complements of A and B, respectively (these are the sets of k for which the corresponding densities are 0). Then

$$f_Z(k) = \begin{cases} 2v_1/x & k \in A \text{ and } k \in B \\ v_1/x & k \in A \text{ and } k \in \overline{B} \\ v_1/x & k \in \overline{A} \text{ and } k \in B \\ 0 & \text{otherwise} \end{cases} \tag{290}$$

Now, the set A is given by

$$A = \left\{ k: \ -\frac{x/2}{v_1} \leq -k \leq 0 \text{ or } 1 - \frac{x}{v_1} \leq k \leq 1 - \frac{x/2}{v_1} \right\} = \left\{ k: \ 0 \leq k \leq \frac{x/2}{v_1} \text{ or } \frac{x/2}{v_1} - 1 \leq k \leq \frac{x}{v_1} - 1 \right\} \tag{291}$$

The inequality $\frac{x/2}{v_1} - 1 \leq k \leq x/v_1 - 1$ is irrelevant since $x < v_1$, implying that $x/v_1 - 1 < 0$ while k must be nonnegative, so

$$A = \left\{ k: \ 0 \leq k \leq \frac{x/2}{v_1} \right\}. \tag{292}$$

Thus, the complement of A is given by

$$\overline{A} = \left\{ k: \ \frac{x/2}{v_1} \leq k \leq 1 \right\}. \tag{293}$$

The set B is given by

$$B = \left\{ k: \ -\frac{x/2}{v_1} \leq 1 - k \leq 0 \text{ or } 1 - \frac{x}{v_1} \leq 1 - k \leq 1 - \frac{x/2}{v_1} \right\} = \left\{ k: \ 1 \leq k \leq 1 + \frac{x/2}{v_1} \text{ or } \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1} \right\} \tag{294}$$

The inequality $1 \leq k \leq 1 + \frac{x/2}{v_1}$ is irrelevant since $k \leq x/v_1 < 1$, so

$$B = \left\{ k: \ \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1} \right\}. \tag{295}$$

The complement of B is given by

$$\overline{B} = \left\{ k: \ 0 \leq k < \frac{x/2}{v_1} \text{ or } \frac{x}{v_1} < k \leq 1 \right\}. \tag{296}$$

Note that A and B are mutually exclusive. The intersection of A and $\overline{B}$ is equal to $\left\{k:\ 0 \leq k \leq \frac{x/2}{v_1}\right\}$. The intersection of $\overline{A}$ and B is equal to $\left\{k : \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1}\right\}$. Thus, from (290), we have that the density of $Z = p_x^{(1)}$ is given by

$$f_Z(k) = v_1/x, \quad 0 \leq k \leq x/v_1, \tag{297}$$

i.e., $p_x^{(1)}$ is distributed Uniform(0, x/v_1).

E.3.2 Distribution of $p_x^{(2)}$

We will follow analogous arguments to those used to derive the distribution of $p_x^{(1)}$ in the previous section. We first obtain the conditional distribution of S given $0 \leq q_x \leq \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} \leq q_x \leq 1$ and use the results we obtained in Appendix E.1 to derive the conditional distribution of $Y = S - \lfloor S \rfloor$. The conditional distribution of S is given by

$$\begin{aligned}
&P(S \leq k \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1) = \\
&P\left(q_x + \tfrac{x/2}{v_1} \leq k \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right) = \\
&P\left(q_x \leq k - \tfrac{x/2}{v_1} \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right) = \\
&P\left(q_x \leq k - \tfrac{x/2}{v_1} \cap \left(0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1\right)\right) / \left(\tfrac{x}{v_1}\right).
\end{aligned} \tag{298}$$

The numerator $P\left(q_x \leq k - \frac{x/2}{v_1} \cap \left(0 \leq q_x \leq \frac{x/2}{v_1} \text{ or } 1 - \frac{x/2}{v_1} \leq q_x \leq 1\right)\right)$ evaluates to

$$\begin{cases}
P\left(0 \leq q_x \leq k - \frac{x/2}{v_1}\right) & 0 \leq k - \frac{x/2}{v_1} < \frac{x/2}{v_1} \\
P\left(0 \leq q_x \leq \frac{x/2}{v_1}\right) & \frac{x/2}{v_1} \leq k - \frac{x/2}{v_1} < 1 - \frac{x/2}{v_1} \\
P\left(1 - \frac{x/2}{v_1} \leq q_x \leq k - \frac{x/2}{v_1}\right) & 1 - \frac{x/2}{v_1} \leq k - \frac{x/2}{v_1} \leq 1
\end{cases} \tag{299}$$

Thus, the conditional distribution function of S given $0 \leq q_x \leq \frac{x/2}{v_1}$ or $1 - \frac{x/2}{v_1} \leq q_x \leq 1$ is equal to

$$\begin{aligned}
&P(S \leq k \mid 0 \leq q_x \leq \tfrac{x/2}{v_1} \text{ or } 1 - \tfrac{x/2}{v_1} \leq q_x \leq 1) = \\
&= \begin{cases}
\left(k - \frac{x/2}{v_1}\right) / \left(\frac{x}{v_1}\right) & \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1} \\
1/2 & \frac{x}{v_1} \leq k < 1 \\
(k-1) / \left(\frac{x}{v_1}\right) & 1 \leq k \leq 1 + \frac{x/2}{v_1}
\end{cases}
\end{aligned} \tag{300}$$

Taking the derivative with respect to k, we have that the density of S is given by

$$f_S(k) = v_1/x, \quad \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1} \text{ or } 1 \leq k \leq 1 + \frac{x/2}{v_1} \tag{301}$$

and 0 otherwise.

Now, using the notation introduced in Appendix E.1, according to (245), we have that the density of $p_x^{(2)} = Y = S - \lfloor S \rfloor$ is equal to

$$f_Y(k) = f_S(k) + f_S(1+k) \tag{302}$$

for $0 \leq k \leq x/v_1$ (recall that the upper bound on Y is now x/v_1 rather than 1, as the entire area covered by the two partially covered voxels is equal to x/v_1, so the area covered by either of the two partially covered voxels cannot exceed this value).

Let A and B denote the sets of values k for which $f_S(k)$ and $f_S(1+k)$, respectively, are nonzero, and let $\overline{A}$ and $\overline{B}$ denote the complements of A and B, respectively (these are the sets of k for which the corresponding densities are 0). Then

$$f_Y(k) = \begin{cases} 2v_1/x & k \in A \text{ and } k \in B \\ v_1/x & k \in A \text{ and } k \in \overline{B} \\ v_1/x & k \in \overline{A} \text{ and } k \in B \\ 0 & \text{otherwise} \end{cases} \tag{303}$$

Now, the set A is given by

$$A = \left\{ k : \ \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1} \text{ or } 1 \leq k \leq 1 + \frac{x/2}{v_1} \right\} \tag{304}$$

The inequality $1 \leq k \leq 1 + \frac{x/2}{v_1}$ is irrelevant since $k \leq x/v_1 < 1$, so

$$A = \left\{ k : \ \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1} \right\}. \tag{305}$$

Thus, the complement of A is given by

$$\overline{A} = \left\{ k : \ 0 \leq k < \frac{x/2}{v_1} \text{ or } \frac{x}{v_1} \leq k \leq 1 \right\}. \tag{306}$$

The set B is given by

$$B = \left\{ k : \ \frac{x/2}{v_1} \leq 1 + k \leq \frac{x}{v_1} \text{ or } 1 \leq 1 + k \leq 1 + \frac{x/2}{v_1} \right\} = \left\{ k : \ \frac{x/2}{v_1} - 1 \leq k \leq \frac{x}{v_1} - 1 \text{ or } 0 \leq k \leq \frac{x/2}{v_1} \right\} \tag{307}$$

The inequality $\frac{x/2}{v_1} - 1 \leq k \leq \frac{x}{v_1} - 1$ is irrelevant since $x < v_1$, implying that $x/v_1 - 1 < 0$ while k must be nonnegative, so

$$B = \left\{ k : \ 0 \leq k \leq \frac{x/2}{v_1} \right\}. \tag{308}$$

The complement of B is given by

$$\overline{B} = \left\{ k : \ \frac{x/2}{v_1} < k \leq 1 \right\}. \tag{309}$$

Note that A and B are mutually exclusive. The intersection of A and $\overline{B}$ is equal to $\left\{k : \frac{x/2}{v_1} \leq k \leq \frac{x}{v_1}\right\}$. The intersection of $\overline{A}$ and B is equal to $\left\{k :\ 0 \leq k \leq \frac{x/2}{v_1}\right\}$. Thus, from (303), we have that the density of $Y = p_x^{(2)}$ is given by

$$f_Y(k) = v_1/x, \qquad 0 \leq k \leq x/v_1, \tag{310}$$

i.e., $p_x^{(2)}$ is also distributed Uniform(0, x/v_1).

Appendix F Distribution of the Product of n Independent Uniform(0, b_i) Random Variables

F.1 General case

Let X_i's, $i = 1, ..., n$, be independently distributed, each with Uniform(0,b_i) distribution, where each b_i is a known real value greater than 0. We want to obtain the distribution of the product of X_i's, i.e., the random variable

$$W = \prod_{i=1}^{n} X_i. \tag{311}$$

Define $T_i = X_i/b_i$ for each $i = 1, ..., n$. Then T_i's are independently and identically distributed as Uniform(0,1) random variables. Let

$$W^* = \prod_{i=1}^{n} T_i = \left(\prod_{i=1}^{n} X_i\right) / \left(\prod_{i=1}^{n} b_i\right) = W / \left(\prod_{i=1}^{n} b_i\right) \tag{312}$$

and $V = -log\ W^*$. Then

$$V = -log\ W^* = -log\left(\prod_{i=1}^{n} T_i\right) = \sum_{i=1}^{n}(-log\ T_i). \tag{313}$$

Now, for $T_i \sim U(0, 1)$, it is a standard result that $-log\ T_i$ has an Exponential(1) distribution [3]. Note that the random variables $-log\ T_i$ are independent from one another since T_i's are independent from one another. Furthermore, it is another standard result that the sum of n independently distributed Exponential(1) random variables is distributed as Gamma(n, 1) [3], so $V \sim$ Gamma$(n, 1)$, that is, the probability density function (pdf) of V is given by

$$g_V(v) = \frac{1}{(n-1)!}\ e^{-v} v^{n-1}, \qquad v > 0 \tag{314}$$

and 0 otherwise. Now, $W^* = e^{-V}$, so its distribution function, denoted by G_W, is given by

$$G_{W^*}(w^*) = P(W^* \leq w^*) = P(e^{-V} \leq w^*) = P(V \geq -log\ w^*) = 1 - G_V(-log\ w^*), \tag{315}$$

where G_V denotes the distribution function of V. Thus, the probability density function of W^* is equal to

$$\begin{aligned} g_{W^*}(w^*) &= \frac{d}{dw^*} G_{W^*}(w^*) = \frac{d}{dw^*}(1 - G_V(-log\ w^*)) = \frac{1}{w^*} \cdot g_V(-log\ w^*) \\ &= \frac{1}{w^*} \cdot \frac{1}{(n-1)!}\ w^*(-log\ w^*)^{n-1} = \frac{(-log\ w^*)^{n-1}}{(n-1)!} \end{aligned} \tag{316}$$

for $0 \leq w^* \leq 1$ and 0 otherwise.

Now, since our variable of interest, W, is related to W^* via

$$W = \left(\prod_{i=1}^{n} b_i\right) \cdot W^*, \tag{317}$$

the distribution function of W can be obtained as follows:

$$G_W(w) = P(W \leq w) = P\left(\left(\prod_{i=1}^{n} b_i\right) \cdot W^* \leq w\right) = P\left(W^* \leq w/\left(\prod_{i=1}^{n} b_i\right)\right) = G_{W^*}\left(w/\left(\prod_{i=1}^{n} b_i\right)\right). \tag{318}$$

Taking a derivative of $G_W(w)$ with respect to w gives the pdf of W:

$$\begin{aligned} g_W(w) &= \frac{d}{dw} G_W(w) = \left(\prod_{i=1}^{n} b_i\right)^{-1} g_{W^*}\left(w/\left(\prod_{i=1}^{n} b_i\right)\right) \\ &= \left(\prod_{i=1}^{n} b_i\right)^{-1} \frac{\left(-log\left(w/\prod_{i=1}^{n} b_i\right)\right)^{n-1}}{(n-1)!} \\ &= \left(\prod_{i=1}^{n} b_i\right)^{-1} \frac{\left(log\left(\prod_{i=1}^{n} b_i\right) - log\ w\right)^{n-1}}{(n-1)!} \end{aligned} \tag{319}$$

for $0 \leq w \leq \prod_{i=1}^{n} b_i$ and 0 otherwise.

While we can obtain the expectation $E(W)$ directly from the density of W by computing the integral

$$E(W) = \int_0^{\prod_{i=1}^{n} b_i} w \cdot g_W(w)\ dw, \tag{320}$$

(this can be done by establishing a recursive relationship in terms of n), it is much easier to use the fact that X_i's are independent and therefore

$$E(W) = E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i) = \prod_{i=1}^{n} \left(\frac{b_i}{2}\right) = \left(\prod_{i=1}^{n} b_i\right)/2^n. \tag{321}$$

Similarly, while the variance can be computed directly from the density of W by first obtaining $E(W^2)$ (also using recursion to obtain the integral for arbitrary n) and then calculating

$Var(W) = E(W^2) - [E(W)]^2$, it is easier to again use the independence of X_i's and X_i^2's:

$$\begin{aligned}
Var(W) &= Var\left(\prod_{i=1}^n X_i\right) = E\left[\left(\prod_{i=1}^n X_i\right)^2\right] - \left[E\left(\prod_{i=1}^n X_i\right)\right]^2 = E\left(\prod_{i=1}^n X_i^2\right) - \left(\prod_{i=1}^n E(X_i)\right)^2 \\
&= \prod_{i=1}^n E(X_i^2) - \left(\prod_{i=1}^n E(X_i)\right)^2 = \prod_{i=1}^n \left(Var(X_i) + (E(X_i))^2\right) - \prod_{i=1}^n (E(X_i))^2 \\
&= \prod_{i=1}^n \left(\frac{b_i^2}{12} + \frac{b_i^2}{4}\right) - \prod_{i=1}^n \frac{b_i^2}{4} = \prod_{i=1}^n \frac{b_i^2}{3} - \prod_{i=1}^n \frac{b_i^2}{4} = \frac{\prod_{i=1}^n b_i^2}{3^n} - \frac{\prod_{i=1}^n b_i^2}{4^n} \\
&= \frac{4^n - 3^n}{12^n} \cdot \prod_{i=1}^n b_i^2. \qquad (322)
\end{aligned}$$

The standard deviation is therefore equal to

$$\sigma_W = \sqrt{Var(W)} = \frac{\sqrt{4^n - 3^n}}{2^n 3^{n/2}} \cdot \prod_{i=1}^n b_i. \qquad (323)$$

Finally, the $100 \times p^{\text{th}}$ percentile of W is the value k such that

$$p = P(W \leq k) = \int_0^k g_W(w)\ dw = \int_0^k \left(\prod_{i=1}^n b_i\right)^{-1} \frac{(log\left(\prod_{i=1}^n b_i\right) - log\ w)^{n-1}}{(n-1)!}\ dw \qquad (324)$$

with $0 \leq p \leq 1$. In particular, the median, or the 50th percentile of W, is the value k so that the integral in (324) evaluates to 0.5. This is easiest to compute numerically.

F.2 Special case: $n = 2$

For $n = 2$, the density of W in (316) becomes

$$g_W(w) = \frac{1}{b_1 b_2} \cdot (log(b_1 b_2) - log\ w), \quad 0 \leq w \leq b_1 b_2 \qquad (325)$$

and 0 otherwise. The probability that W is in the interval (A, B) with $0 \leq A < B \leq b_1 b_2$ is therefore equal to

$$\begin{aligned}
P(A \leq W \leq B) &= \int_A^B (b_1 b_2)^{-1} (log(b_1 b_2) - log\ w)\ dw \\
&= \frac{1}{b_1 b_2} \left[(log(b_1 b_2)(B - A) + w(1 - log\ w)\ |_A^B\right] \\
&= \frac{1}{b_1 b_2} \cdot (log(b_1 b_2) + 1) \cdot (B - A)\ +\ \frac{1}{b_1 b_2} \cdot (A\ log\ A - B\ log\ B). \qquad (326)
\end{aligned}$$

with $0 \cdot log\ 0 \equiv 0$.

When $b_1 = b_2 = 1$, the density of W is then equal to

$$g_W(w) = -log\ w, \quad 0 \leq w \leq 1 \tag{327}$$

and 0 otherwise. In this special case, the probability that W is in the interval (A, B) with $0 \leq A < B \leq 1$ is hence equal to

$$P(A \leq W \leq B) = B(1 - log\ B) - A(1 - log\ A) \tag{328}$$

with $0 \cdot log\ 0 \equiv 0$.

Also, according to (321) and (323), the expectation $E(W) = (b_1 b_2)/4$ and the standard deviation $\sigma_W = \frac{\sqrt{16-9}}{(4 \cdot 3} \cdot (b_1 b_2) = \frac{\sqrt{7}}{12} \cdot (b_1 b_2) = 0.2205 \cdot (b_1 b_2)$. In the special case when $b_1 = b_2 = 1$, the expectation and standard deviation are then equal to 1/4 and 0.2205, respectively.

The median of W is the value k that solves the equation

$$0.5 = P(W \leq k) = k(log(b_1 b_2) + 1 - log\ k)/(b_1 b_2). \tag{329}$$

When $b_1 = b_2 = 1$, the solution is $k = 0.1867$. Thus, the solution for arbitrary values of b_1 and b_2 is $k \cdot (b_1 b_2) = 0.1867 \cdot (b_1 b_2)$.

F.3 Special case: $n = 3$

For $n = 3$, the density of W in (316) becomes

$$g_W(w) = \frac{(log(b_1 b_2 b_3) - log\ w)^2}{2(b_1 b_2 b_3)}, \quad 0 \leq w \leq b_1 b_2 b_3 \tag{330}$$

and 0 otherwise. When $b_1 = b_2 = b_3 = 1$, the density of W is given by

$$g_W(w) = \frac{(log\ w)^2}{2}, \quad 0 \leq w \leq 1 \tag{331}$$

and 0 otherwise. In this special case, the probability that W is in the interval (A, B) with with $0 \leq A < B \leq 1$ is then equal to

$$\begin{aligned} P(A \leq W \leq B) &= \int_A^B \frac{(log\ w)^2}{2}\ dw = \frac{w}{2}\left((log\ w - 1)^2 + 1\right)\Big|_A^B \\ &= \frac{B\left((log\ B - 1)^2 + 1\right) - A\left((log\ A - 1)^2 + 1\right)}{2} \end{aligned} \tag{332}$$

Note that the expression in (332) is equal to the probability $P(A \leq W^* \leq B)$, with W^* defined in (312), since W^* is the product of Uniform(0,1) random variables. Moreover, from

(312), we also have that $W = (b_1b_2b_3)W^*$. Therefore, using the expression in (332), we can get the probability $P(A \leq W \leq B)$ for arbitrary b_1, b_2 and b_3, with $0 \leq A < B \leq b_1b_2b_3$, as follows:

$$\begin{aligned} P(A \leq W \leq B) &= P(A \leq (b_1b_2b_3)W^* \leq B) = P\left(\frac{A}{b_1b_2b_3} \leq W^* \leq \frac{B}{b_1b_2b_3}\right) \\ &= \frac{\frac{B}{b_1b_2b_3}\left((log(B/b_1b_2b_3)-1)^2+1\right) - \frac{A}{b_1b_2b_3}\left((log(A/b_1b_2b_3-1)^2+1\right)}{2} \\ &= \frac{B\left((log\ B - log(b_1b_2b_3)-1)^2+1\right) - A\left((log\ A - log(b_1b_2b_3)-1)^2+1\right)}{2(b_1b_2b_3)} \end{aligned} \tag{333}$$

Also, according to (321) and (323), the expectation $E(W) = (b_1b_2b_3)/8$ and the standard deviation $\sigma_W = \frac{\sqrt{(64-27}}{(2^3\sqrt{27}} \cdot (b_1b_2b_3) = 0.1463 \cdot (b_1b_2b_3)$. The median of W is the value k that solves the equation

$$0.5 = P(W \leq k) = \frac{k}{2(b_1b_2b_3)}\left((log\ k - log(b_1b_2b_3)-1)^2+1\right), \tag{334}$$

For $b_1 = b_2 = b_3 = 1$, the solution is $k = 0.069$. The solution for arbitrary values of b_1, b_2 and b_3 is therefore $k \cdot (b_1b_2b_3) = 0.069 \cdot (b_1b_2b_3)$.

F.4 Distribution of a Positive Constant Times the Product of Two Independent Uniform(0, b_i) Random Variables

In this section, we derive the distribution of the random variable $Y = cX_1X_2$ where c is a positive constant and X_1 and X_2 are independently distributed Uniform(0, b_1) and Uniform(0, b_2), respectively. In Appendix F.2, we obtained the distribution of $W = X_1X_2$, given in (325). Now, to obtain the distribution of $Y = cW$, we note that the distribution function of Y is given by

$$G_Y(y) = P(Y \leq y) = P(cW \leq y) = P(W \leq y/c) = G_W(y/c), \tag{335}$$

so the pdf of Y is given by

$$g_Y(y) = \frac{d}{dy}G_Y(y) = \frac{1}{c} \cdot g_W(y/c). \tag{336}$$

Putting together (336) and (325), we have that the pdf of Y is given by

$$g_Y(y) = \frac{1}{cb_1b_2} \cdot (log(cb_1b_2) - log\ y), \quad 0 \leq y \leq cb_1b_2 \tag{337}$$

and 0 otherwise. We can also use the result in (326) to get the expression for $P(A \leq Y \leq B)$ with $0 \leq A < B \leq cb_1b_2$:

$$\begin{aligned} P(A \leq Y \leq B) &= P(A \leq cW \leq B) = P(A/c \leq W \leq B/c) \\ &= \frac{1}{cb_1b_2} \cdot (log(cb_1b_2) + 1) \cdot (B - A) \ + \ \frac{1}{cb_1b_2} \cdot (A \ log \ A - B \ log \ B) \end{aligned} \tag{338}$$

with $0 \cdot log \ 0 \equiv 0$, as before.

From the results for $E(W)$ and σ_W in Appendix F.2, we have that the expected value of Y is equal to

$$E(Y) = E(cW) = cE(W) = (cb_1b_2)/4, \tag{339}$$

the median is equal to $0.1867 \cdot (cb_1b_2)$, and the standard deviation of Y is equal to

$$\sigma_Y = \sigma_{cW} = c\sigma_W = 0.2205 \cdot (cb_1b_2). \tag{340}$$

Appendix G Variance of N_p in 3-Dimensional Setting (No Small Dimensions)

In 3-dimensional setting, when $x \geq v_1$, $y \geq v_2$ and $z \geq v_3$ (i.e., no small dimensions), the number of partially covered voxels N_p is given by

$$N_p = 8 + 4(f_x + f_y + f_z) + 2(f_x f_y + f_x f_z + f_y f_z). \tag{341}$$

Therefore, the variance of N_p is equal to

$$\begin{aligned} Var(N_p) &= 16\left[Var(f_x) + Var(f_y) + Var(f_z)\right] + 4\left[Var(f_x f_y) + Var(f_x f_y) + Var(f_y f_z)\right] && (342) \\ &+ 16\left[C(f_x, f_x f_y) + C(f_x, f_x f_z) + C(f_y, f_x f_y) + C(f_y, f_y f_z) + C(f_z, f_x f_z) + C(f_z, f_y f_z)\right] \\ &+ 8\left[C(f_x f_y, f_x f_z) + C(f_x f_y, f_y f_z) + C(f_x f_z, f_y f_z)\right], && (343) \end{aligned}$$

where $C(A, B)$ denotes the covariance of the random variables A and B (we have to account for several covariances because several of the summands in (341) are not independent).

The variances of f_x, f_y and f_z were derived in (25), (39) and (89), respectively. The variance of the product $f_x f_y$ was derived in (46) in Section 2.2.1. The expressions for the variances of $f_x f_z$ and $f_y f_z$ are exactly analogous. Using the independence of f_x and f_y, we have the covariance $C(f_x, f_x f_y)$ given by

$$\begin{aligned} C(f_x, f_x f_y) &= E(f_x^2 f_y) - E(f_x)E(f_x f_y) = E(f_x^2)E(f_y) - E(f_x)E(f_x)E(f_y) \\ &= \left(E(f_x^2) - (E(f_x))^2\right) E(f_y) = Var(f_x)E(f_y). \end{aligned} \tag{344}$$

The expressions for the other 5 covariances of one of f_x, f_y or f_z with a pair of these are analogous. The covariance of $f_x f_y$ and $f_x f_z$ is given by

$$\begin{aligned} C(f_x f_y, f_x f_z) &= E(f_x^2 f_y f_z) - E(f_x f_y)E(f_x f_z) = E(f_x^2)E(f_y)E(f_z) - E(f_x)E(f_y)E(f_x)E(f_z) \\ &= \left(E(f_x^2) - (E(f_x))^2\right) E(f_y)E(f_z) = Var(f_x)E(f_y)E(f_z). \end{aligned} \tag{345}$$

The covariances of the other two sets of pairs are analogous. Putting together everything

together, we have that the variance of N_p is given by

$$
\begin{aligned}
Var(N_p) &= 16\left(Var(f_x)+Var(f_y)+Var(f_z)\right) \\
&+ 4\left(Var(f_xf_y)+Var(f_xf_z)+Var(f_yf_z)\right) \\
&+ 8\left(Var(f_x)E(f_y)E(f_z)+Var(f_y)E(f_x)E(f_z)+Var(f_z)E(f_x)E(f_y)\right) \\
&+ 16\left(Var(f_x)(E(f_y)+E(f_z))+Var(f_y)(E(f_x)+E(f_z))+Var(f_z)(E(f_x)+E(f_y))\right) \\
&= 4\left(Var(f_xf_y)+Var(f_xf_z)+Var(f_yf_z)\right) \\
&+ 8\left[Var(f_x)(2E(f_y)+2E(f_z)+E(f_y)E(f_z)+2)\right. \\
&+ Var(f_y)(2E(f_x)+2E(f_z)+E(f_x)E(f_z)+2) \\
&+ \left.Var(f_z)(2E(f_x)+2E(f_y)+E(f_x)E(f_z)+2)\right] \\
&= 4\left(Var(f_xf_y)+Var(f_xf_z)+Var(f_yf_z)\right) \\
&+ 8\left[Var(f_x)\left(\frac{y}{v_2}\frac{z}{v_3}+\frac{y}{v_2}+\frac{z}{v_3}-1\right)+Var(f_y)\left(\frac{x}{v_1}\frac{z}{v_3}+\frac{x}{v_1}+\frac{z}{v_3}-1\right)\right. \\
&+ \left.Var(f_z)\left(\frac{x}{v_1}\frac{y}{v_2}+\frac{x}{v_1}+\frac{y}{v_2}-1\right)\right].
\end{aligned}
\tag{346}
$$

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