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The Bells' Capture note TH-3054-CERN

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This document revisits the paper by M. Bell and J. S. Bell "Capture of Cooling Electrons by Cool Protons" TH-3054-CERN (March 30, 1981). I expand the treatment to include e^+e^- capture.

First Approximation

The capture cross section of a non-relativistic electron (velocity v_e) on a stationary nucleus, from Spitzer [*Physics of Fully Ionized Gases* second revised edition, Eq. (5-62)]:

$$\sigma(v_e) = A \sum_{n=1}^{\infty} \frac{\nu_0}{\nu_n} \frac{h\nu_0}{E} \frac{g_n}{n^3} \tag{1}$$

where:

A	$=$	$2^4 3^{-3/2} he^2/(m_e^2 c^3) = 2.11 \times 10^{-22}$ cm^2 [Spitzer, Eq. (5-63)]:
h	$=$	Planck's constant
e	$=$	electron charge
m_e	$=$	electron mass
c	$=$	velocity of light
$h\nu_0$	$=$	$Z^2\alpha^2 m_e c^2/2$
	$=$	ground state binding energy
	$=$	13.6 eV for hydrogen
$h\nu_n$	$=$	radiated photon energy in capture to level n
	$=$	$h\nu_0/n^2 + E$
E	$=$	electron kinetic energy
	$=$	$m_e v_e^2/2$
α	$=$	fine structure constant
	$=$	$1/137$

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and g_n "is a correction factor, generally about equal to one." For the purpose of this approximation:

$$g_n = 1 \tag{2}$$

Re-writing Eq. 1:

$$\sigma = A \left(\frac{h\nu_0}{E} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n\,(n^2 + h\nu_0/E)} \tag{3}$$

Recognizing [*e.g.* from Abramowitz and Stegun (6.3.17)] that:

$$\Re\psi(1+iy) = -\gamma + y^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+y^2)}$$

for $-\infty < y < \infty$, leads to the expression:

$$\sigma = \frac{A}{2} \frac{h\nu_0}{E} \left\{ \psi \left(1 + i\sqrt{\frac{h\nu_0}{E}} \right) + \psi \left(1 - i\sqrt{\frac{h\nu_0}{E}} \right) + 2\gamma_1 \right\} \tag{4}$$

where ψ is the digamma function:

$$\psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}$$

and γ_1 is Euler's constant:

$$\gamma_1 = \gamma = 0.5772156649\ldots \tag{5}$$

The asymptotic expression for $\psi(y)$ is [*e.g.* from Abramowitz and Stegun (6.3.19)] as:

$$\Re\psi(1+iy) = \ln y + \frac{1}{12y^2} + \frac{1}{120y^4} + \frac{1}{252y^6} + \cdots$$

Using this form of ψ re-write Eq. 4 as:

$$\sigma = A\frac{h\nu_0}{E} \left[\gamma_1 + \ln\sqrt{\frac{h\nu_0}{E}} \right] \tag{6}$$

ignoring terms of order $E/h\nu_0$ on the right of Eq. 6. This is accurate where:

$$\frac{E}{h\nu_0} << 1 \tag{7}$$

For example with $E/h\nu_0 = 0.2$, the term $1/12y^2 = 1/300$ compared to $\ln y = 0.8$.

The rate-of-capture per proton is:

$$\alpha_r n_e \tag{8}$$

where:

n_e = number of electrons per unit volume

α_r = $\langle v_e \sigma(v_e) \rangle$

where the angular brackets denote averaging over the electron velocity distribution.

Taking the electron velocity to be equal to a characteristic "thermal" velocity, $v_e = v_T$ re-write Eq. 6:

$$\alpha_r = AZ^2\alpha^2c^2v_T^{-1}\left[\gamma_1 + \ln\left(\frac{Z\alpha c}{v_T}\right)\right] \tag{9}$$

for v_T small the ln term dominates and:

$$\alpha_r \approx 19.4 Z^2 \alpha r_e^2 c^2 v_T^{-1} \ln\left(\frac{Z\alpha c}{v_T}\right) \tag{10}$$

where $r_e = e^2/(m_e c^2)$ is the classical radius of the electron.

Second Approximation

Taking the electron velocity distribution as a Maxwellian:

$$f(\overrightarrow{v_e}) = \left(\frac{2\pi kT}{m_e}\right)^{-3/2} e^{-E/kT} \tag{11}$$

or the "flattened" Maxwellian:

$$f(\overrightarrow{v_e}) = \left(\frac{2\pi kT}{m_e}\right)^{-1} e^{-E/kT}\delta(v_{ex}) \tag{12}$$

suppressing the x degree-of-freedom (for example). Then:

$$\begin{aligned}\alpha_r &= \langle v_e \sigma(v_e) \rangle \\ &= \frac{\int_0^\infty dv_e v_e^{n-1} v_e \sigma(v_e) e^{-E/kT}}{\int_0^\infty dv_e v^{n-1} e^{-E/kT}}\end{aligned} \tag{13}$$

where $n = 3$ for the Maxwellian, and $n = 2$ for the "flattened" Maxwellian.

Taking a change of variables:

$$z = \frac{E}{kT} \tag{14}$$

and substituting this into Eq. 13 and Eq. 6:

$$\begin{aligned}\alpha_r &= h\nu_0 A\sqrt{\frac{2}{m_e kT}}\frac{\int_0^\infty dz z^{\frac{n-3}{2}}[\gamma_1 + \frac{1}{2}\ln(\frac{h\nu_0}{kT}) - \frac{1}{2}\ln(z)]e^{-z}}{\int_0^\infty dz z^{\frac{n-2}{2}}e^{-z}} \\ &= h\nu_0 A\sqrt{\frac{2}{m_e kT}}\frac{\{[\gamma_1 + \frac{1}{2}\ln(\frac{h\nu_0}{kT})]\Gamma(\frac{n-2}{2}) - \frac{1}{2}\int_0^\infty dz z^{\frac{n-3}{2}}e^{-z}\ln(z)\}}{\Gamma(\frac{n}{2})}\end{aligned} \tag{15}$$

There is the need to perform the integration:

$$\int_0^\infty dz\, z^{\frac{n-3}{2}} e^{-z}\ln(z) \approx \int_0^\infty dz\, z^{\frac{n-3}{2}} e^{-z}\psi(z)$$

where the right hand side is written in terms of the ψ function. Recall the definition of the Γ function [*e.g.* from Abramowitz and Stegun (6.1.1)]:

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1}e^{-t}$$

the integral of interest could be written:

$$\int_0^\infty dz\, z^{\frac{n-3}{2}} e^{-z}\psi(z) = \int d\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)\frac{d\ln(\Gamma(z))}{dz} = \frac{d}{dz}\int d\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)\ln(\Gamma(z))$$

now doing the integral in the straight forward manner (assuming that the integral over $0 \le z \le \infty$ picks out the value $z = \frac{n}{2} - \frac{1}{2}$, but resisting from immediately substituting):

$$\frac{d}{dz}\int d\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)\ln(\Gamma(z)) = \left[\ln(\Gamma(z))\frac{d\Gamma(z)}{dz} + \Gamma(z)\frac{d\ln(\Gamma(z))}{dz} - \frac{d\Gamma(z)}{dz}\right]_{z=\frac{n}{2}-\frac{1}{2}}$$

recognize the second term of the right hand side to be $\Gamma(z)\psi(z)$ and:

$$\ln(\Gamma(\frac{n}{2}-\frac{1}{2}))\frac{d\Gamma(\frac{n}{2}-\frac{1}{2})}{dz}+\Gamma(\frac{n}{2}-\frac{1}{2})\psi(\frac{n}{2}-\frac{1}{2})-\frac{d\Gamma(\frac{n}{2}-\frac{1}{2})}{dz} = \Gamma(\frac{n}{2}-\frac{1}{2})\psi(\frac{n}{2}-\frac{1}{2})$$

recognizing that the derivatives on the left hand side vanish (no dependence on z) This allows us to write:

$$\alpha_r = h\nu_0 A\sqrt{\frac{2}{m_e kT}}\left[\gamma_1 + \frac{1}{2}\ln(\frac{h\nu_0}{kT}) - \frac{1}{2}\psi(\frac{n}{2} - \frac{1}{2})\right]\frac{\Gamma(\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{n}{2})} \tag{16}$$

Now substituting the values for the "Maxwell" and the "flattened" thermal distributions ($n = 3,\ 2$ respectively), and using the values:

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \Gamma(1) = 1, \quad \Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}, \quad \psi(\tfrac{1}{2}) = -\gamma - 2\ln 2, \quad \psi(1) = -\gamma$$

we have:

$$\alpha_r(\text{Maxwell}) = \frac{\sqrt{2}ch\nu_0 A}{\sqrt{m_e c^2\ kT}}\frac{2}{\sqrt{\pi}}\left[\gamma_1 + \frac{1}{2}\gamma + \frac{1}{2}\ln(h\nu_0/kT)\right] \tag{17}$$

$$\alpha_r(\text{flattened}) = \frac{\sqrt{2}ch\nu_0 A}{\sqrt{m_e c^2\ kT}}\sqrt{\pi}\left[\gamma_1 + \frac{1}{2}\gamma + \frac{1}{2}\ln(h\nu_0/kT) + \ln(2)\right] \tag{18}$$

This result is compared with Spitzer's Eq. (5-67):

$$\alpha_r = 2A\sqrt{\frac{2kT}{\pi m_e}}\beta\phi(\beta) \tag{19}$$

where:

$$\beta = \frac{h\nu_0}{kT} \tag{20}$$

Spitzer's definition of $\phi(\beta)$ is his Eq. (5-69):

$$\phi(\beta) = \sum_{n=1}^{\infty}\frac{\beta}{n^3}e^{\beta/n^2}\left[-\mathrm{Ei}\left(-\frac{\beta}{n^2}\right)\right]$$

associating this with above approximation taking $\gamma_1 = \gamma$:

$$\phi(\beta) = \frac{3}{2}\gamma + \ln(\sqrt{\beta}) \tag{21}$$

$$= \ln(2.377\sqrt{\beta}) \tag{22}$$

which can be compared with Spitzer's Table 5-6.

β	0.01	0.02	0.05	0.1	0.2	0.5
$\phi(\beta)$ Eq. 5-69	0.053	0.09	0.18	0.28	0.43	0.70
$\phi(\beta)$ Eq. 22	-0.603	-0.34	0.01	0.23	0.44	0.69
β	1	2	5	10	100	1000
$\phi(\beta)$ Eq. 5-69	0.96	1.26	1.69	2.02	3.2	4.3
$\phi(\beta)$ Eq. 22	0.85	0.98	1.13	1.22	1.4	1.6

Table 1: Comparison of $\phi(\beta)$ from Eq. 5-69 and Eq. 22 from Spitzer's Table 5-6.

Third Approximation

The Bells go on to compare the cross section Eq. 6 with those computed in Bates, *et al.*, "Dissociation, recombination and attachment processes in the upper atmosphere; II. The rate of recombination," *Proc. R. Soc. Lond. A* **170**, 322 (1939), in particular using Table I from that work. These results are contained in Eq. (15) of that work:

$$Q_e^n = q_u^n + q_e^n$$

being the recombination cross section to the state n, where:

$$q_u^n = A_B \frac{32n^2\lambda^{10}}{(n^2+\lambda^2)^4} \sum_{s=0}^{n-2} (s+1)(n-s-1) \left| F\left(1-i\lambda, 2+s-n; 2; \frac{-4in\lambda}{(n-i\lambda)^2}\right)\right|^2$$

$$\begin{aligned} q_z^n &= A_B \tfrac{\lambda^6}{(n^2+\lambda^2)^2} \textstyle\sum_{s=0}^{n-1} \Big|(n-s-1)F\left(-i\lambda, 2+s-n; 1; \tfrac{-4in\lambda}{(n-i\lambda)^2}\right) \\ &- \left(n-s-1-\tfrac{2in\lambda}{n+i\lambda}\right) F\left(-i\lambda, 1+s-n; 1; \tfrac{-4in\lambda}{(n-i\lambda)^2}\right)\Big|^2 \end{aligned}$$

$$A_B = \frac{8}{3\pi}\frac{h^4}{m_e^3 e^2}\frac{\nu}{c^3}\mathrm{cosech}(\pi\lambda) e^{\pi\lambda - 4\lambda \arctan(n/\lambda)}$$

$$\frac{8}{3\pi}\frac{h^4}{m_e^3 e^2}\frac{\nu}{c^3} = 1.146 \times 10^{-21}\mathrm{cm}^2$$

$$\lambda = \frac{1}{\sqrt{2\epsilon}} = \frac{2\pi e^2}{hv} = \frac{\alpha}{\beta_e}$$

with ϵ the electron energy and α here being the fine-structure constant. The "total" recombination cross section is then:

$$\sum_{n=1}^{\infty} Q_e^n$$

in practice the upper limit of the sum is set to the desired level of precision. This result is attributed to J. R. Oppenheimer "Über die Strahlung der freien Elektronen im Coulombfeld", *Zeitschrift für Physik* **55** 725 (1929).

The function $F(a, b; c; z)$ is the hypergeometric function. We make use of the property [see *e.g.* from Abramowitz and Stegun (15.4.1)]:

$$F(-m, b; c; z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

which is allows the function to be calculated as a finite series in the expressions for q_u^n and q_z^n above. This condition prevails above (where $b = -m$ and $m = 0, 1, 2, \ldots$). A modern recalculation of the Bates, *et al.* result achieves

$h\nu_0/E$	49.0	100	196	400
E (Z=1, eV)	0.278	0.136	0.69	0.34
σ Bates, *et al.*	23.0	53.7	119	272
	25.1	59.6	133	308
σ Eq. 6	26.1	60.9	133	302
σ Eq. 23	23.1	53.9	119	271

Table 2: Comparison of cross sections for various calculations. The line under the Bates *et al.* is a recalculation of their results.

a somewhat different set of values than presented in that paper. This difference seems to be due to the difference in the principle quantum number, n, dependence. These values agree with Eq. 6. Better agreement with the original Bates, *et al.* result uses the prescription due to M. J. Seaton "The Solution of Capture-Cascade Equations for Hydrogen", *Monthly Notes of the Royal Astronomical Society*, **119**, 81 (1959):

$$\sigma = A \frac{h\nu_0}{E} \left[\ln \left(\sqrt{\frac{h\nu_0}{E}} \right) + \gamma_1 + \gamma_2 \left(\frac{E}{h\nu_0} \right)^{1/3} \right] \tag{23}$$

where

$$\gamma_1 = 0.1402 \qquad \gamma_2 = 0.525$$

The three expressions for the recombination cross sections, along with the original Bates *et al.* calculations are shown in Fig. 1. Integrating Eq. 23 over

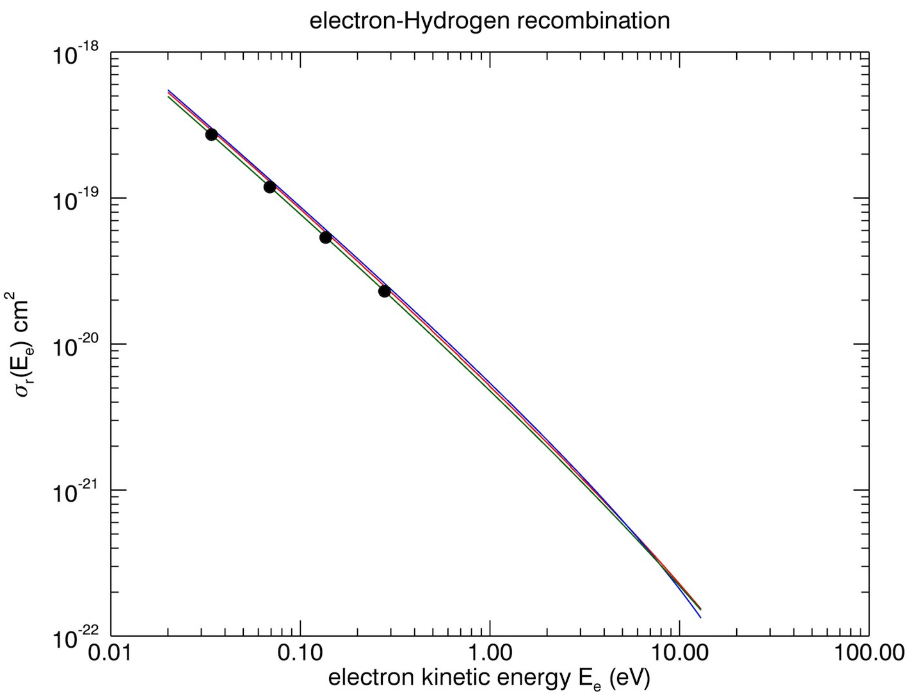

Figure 1: The comparison of recombination cross sections as a function of electron kinetic energy for Eqs. 6 (blue) and 23 (green) and the Bates, *et al.* values (black symbols) and recalculation (red).

the electron temperature distributions we arrive at the equivalent of Eq. 16:

$$\left[\ln\sqrt{\frac{h\nu_0}{kT}} - \frac{1}{2}\psi(\frac{n}{2}-\frac{1}{2}) + \gamma_1 + \gamma_2\frac{\Gamma(\frac{n}{2}-\frac{1}{6})}{\Gamma(\frac{n}{2}-\frac{1}{2})}\left(\frac{kT}{h\nu_0}\right)^{1/3}\right]\frac{\Gamma(\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{n}{2})} \tag{24}$$

and the subsequent expressions:

$$\alpha_r(\mathrm{Max.}) = \frac{\sqrt{2}ch\nu_0 A}{\sqrt{m_e c^2\, kT}}\frac{2}{\sqrt{\pi}}\left[\gamma_1 + \frac{1}{2}\gamma + \Gamma(\frac{4}{3})\gamma_2\left(\frac{kT}{h\nu_0}\right)^{\frac{1}{3}} + \ln\sqrt{\frac{h\nu_0}{kT}}\right] \tag{25}$$

$$\alpha_r(\mathrm{flat.}) = \frac{\sqrt{2}ch\nu_0 A}{\sqrt{m_e c^2\, kT}}\sqrt{\pi}\left[\gamma_1 + \frac{1}{2}\gamma + \Gamma(\frac{5}{6})\gamma_2\left(\frac{kT}{h\nu_0}\right)^{\frac{1}{3}} + \ln 2\sqrt{\frac{h\nu_0}{kT}}\right] \tag{26}$$

with kT in eV and Z=1:

$$\alpha_r(\mathrm{Max.}) = \frac{1.92}{\sqrt{kT}}\left[\ln\frac{5.66}{\sqrt{kT}} + 0.196(kT)^{\frac{1}{3}}\right]\ 10^{-13}\ \mathrm{cm}^3/\mathrm{s} \tag{27}$$

$$\alpha_r(\mathrm{flat.}) = \frac{3.02}{\sqrt{kT}}\left[\ln\frac{11.3}{\sqrt{kT}} + 0.140(kT)^{\frac{1}{3}}\right]\ 10^{-13}\ \mathrm{cm}^3/\mathrm{s} \tag{28}$$

Note that Eqs. 17 and 18 can also be written as:

$$\alpha_r(\mathrm{Max.}) = \frac{1.92}{\sqrt{kT}}\ln\left(\frac{5.66}{\sqrt{kT}}\right)\ 10^{-13}\ \mathrm{cm}^3/\mathrm{s}$$

$$\alpha_r(\mathrm{flat.}) = \frac{3.02}{\sqrt{kT}}\ln\left(\frac{11.3}{\sqrt{kT}}\right)\ 10^{-13}\ \mathrm{cm}^3/\mathrm{s}$$

The Spitzer form, Eq. 19 can be written:

$$\alpha_r(\mathrm{Spitz.}) = \frac{1.92}{\sqrt{kT}}\ln\left(\frac{8.77}{\sqrt{kT}}\right)\ 10^{-13}\ \mathrm{cm}^3/\mathrm{s}$$

Eqs. 27 and 28 are further simplified to:

$$\alpha_r(\mathrm{Max.}) = 3.78(kT)^{-0.682}\ 10^{-13}\ \mathrm{cm}^3/\mathrm{s} \tag{29}$$

$$\alpha_r(\mathrm{flat.}) = 7.86(kT)^{-0.648}\ 10^{-13}\ \mathrm{cm}^3/\mathrm{s} \tag{30}$$

agreeing within 10% with the more complex equations in the range: $0.01 < kT < 3.0$ eV, and to within 1% for $0.07 < kT < 0.7$ eV. The other expressions have a more limited range of precision:

$$\begin{aligned} \alpha_r(\text{Max. Eq.17}) &= 3.45(kT)^{-0.710}\ 10^{-13}\ \text{cm}^3/\text{s} & (31)\\ \alpha_r(\text{flat. Eq. 18}) &= 7.48(kT)^{-0.662}\ 10^{-13}\ \text{cm}^3/\text{s} & (32)\\ \alpha_r(\text{Spitz.}) &= 4.28(kT)^{-0.677}\ 10^{-13}\ \text{cm}^3/\text{s} & (33) \end{aligned}$$

agreeing within 10% with the more complex equations in the range: $0.02 < kT < 1.8$ eV, and to within 1% for $0.10 < kT < 0.66$ eV. These various forms

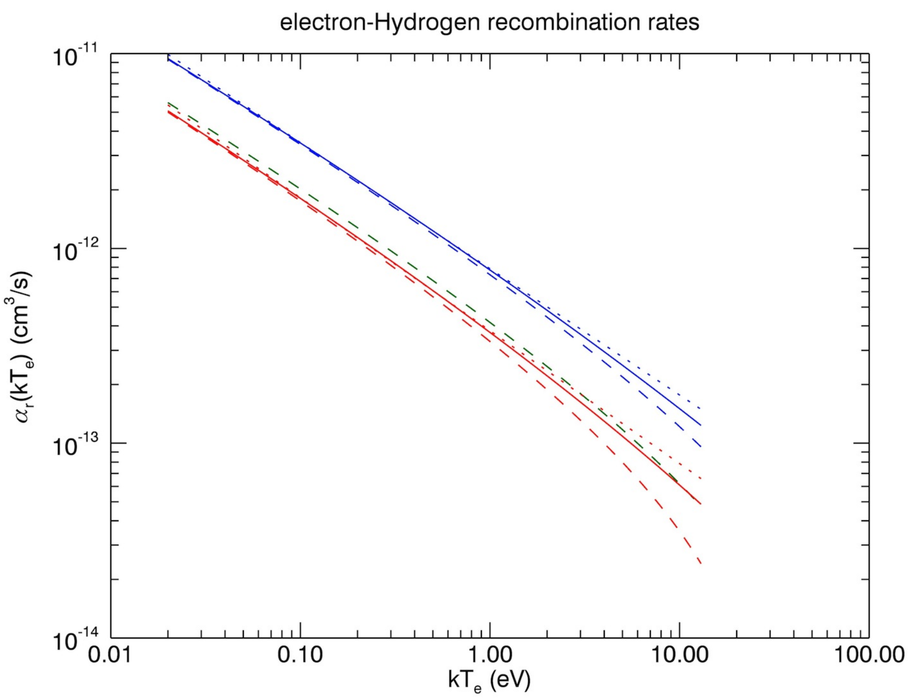

Figure 2: The comparison of recombination rates as a function of electron temperature for Eqs. 17 (red dashed) and 27 (red solid) and the Spitzer values (green dashed) which are averaged over the Maxwell distribution. The values from Eqs. 18 (blue dashed) and 28 are averaged over the "flattened" distributions. The dotted lines are the approximations Eqs. 29 and 30 to be compared with the solid lines.

are compared in Fig. 2. The effect of a "flattened" thermal distribution increases the recombination rates by roughly a factor of 2. The different recombination cross sections do not have a large effect on the recombination rate below $kT = 1$ eV.

Number of reactions in the lab frame

The rate of recombinations in the electron-Hydrogen "plasma" is given by:

$$N_r' = \alpha_r n_e' d \tag{34}$$

where n_e' is the density of electrons *in the plasma reference frame,* and d is the fraction of the ion orbit that overlaps with the electrons. In the plasma frame, the electron number density is related to the lab-frame quantity by: $n_e' = n_e/\gamma$ where γ is the boost into the lab frame. The times are also dilated boosting to the lab frame, to longer times. This results in the expression:

$$N_r = \alpha_r(kT) n_e d \frac{1}{\gamma^2} \tag{35}$$

where the electron temperature has its *plasma frame* value. Note that d is frame invariant.

$$\gamma = \frac{1}{\sqrt{1-(v_e/c)^2}} \tag{36}$$

where v_e is the mean electron velocity in the lab frame (usually equal to the ion velocity).

Protonium and positronium

The Appendix considers the case:

$$p + \bar{p} \to (p\bar{p}) + \gamma \tag{37}$$

where $(p\bar{p})$ is the bound state of a proton and an anti-proton ("protonium"). In this treatment I'll include:

$$e^+ + e^- \to \left(e^+ e^-\right)_{\mathrm{Ps}} + \gamma \tag{38}$$

where $(e^+e^-)_{\mathrm{Ps}}$ is the bound state of the electron and positron, positronium.

The changes to the above formula take the electron mass m_e and replace it with the appropriate reduced mass:

$$m_{p\bar{p}} = m_p/2 \tag{39}$$
$$m_{e^+e^-} = m_e/2 \tag{40}$$

The rate equations 17, 18, 19, 25, and 26 become for $p\bar{p}$:

$$\alpha_r(\text{Max.}) = \frac{1.38}{\sqrt{kT}} \ln\left(\frac{243}{\sqrt{kT}}\right)\ 10^{-17}\ \text{cm}^3/\text{s} \tag{41}$$
$$\alpha_r(\text{flat.}) = \frac{2.17}{\sqrt{kT}} \ln\left(\frac{485}{\sqrt{kT}}\right)\ 10^{-17}\ \text{cm}^3/\text{s} \tag{42}$$
$$\alpha_r(\text{Spitz.}) = \frac{1.38}{\sqrt{kT}} \ln\left(\frac{376}{\sqrt{kT}}\right)\ 10^{-17}\ \text{cm}^3/\text{s} \tag{43}$$
$$\alpha_r(\text{Max.}) = \frac{1.38}{\sqrt{kT}} \left[\ln\frac{243}{\sqrt{kT}} + 0.016(kT)^{\frac{1}{3}}\right]\ 10^{-17}\ \text{cm}^3/\text{s} \tag{44}$$
$$\alpha_r(\text{flat.}) = \frac{2.17}{\sqrt{kT}} \left[\ln\frac{485}{\sqrt{kT}} + 0.011(kT)^{\frac{1}{3}}\right]\ 10^{-17}\ \text{cm}^3/\text{s} \tag{45}$$

and for e^+e^-:

$$\alpha_r(\text{Max.}) = \frac{5.42}{\sqrt{kT}} \ln\left(\frac{4.00}{\sqrt{kT}}\right)\ 10^{-13}\ \text{cm}^3/\text{s} \tag{46}$$
$$\alpha_r(\text{flat.}) = \frac{8.53}{\sqrt{kT}} \ln\left(\frac{8.01}{\sqrt{kT}}\right)\ 10^{-13}\ \text{cm}^3/\text{s} \tag{47}$$
$$\alpha_r(\text{Spitz.}) = \frac{5.42}{\sqrt{kT}} \ln\left(\frac{6.20}{\sqrt{kT}}\right)\ 10^{-13}\ \text{cm}^3/\text{s} \tag{48}$$
$$\alpha_r(\text{Max.}) = \frac{5.42}{\sqrt{kT}} \left[\ln\frac{4.00}{\sqrt{kT}} + 0.247(kT)^{\frac{1}{3}}\right]\ 10^{-13}\ \text{cm}^3/\text{s} \tag{49}$$
$$\alpha_r(\text{flat.}) = \frac{8.52}{\sqrt{kT}} \left[\ln\frac{8.01}{\sqrt{kT}} + 0.174(kT)^{\frac{1}{3}}\right]\ 10^{-13}\ \text{cm}^3/\text{s} \tag{50}$$

The temperature is taken in the plasma-reference frame. The reactions rates will be roughly four orders of magnitude lower for ($p\bar{p}$) production and a factor of two higher for $(e^+e^-)_{\text{Ps}}$ production relative to the (pe^-) production. These rates are shown as a function of temperature in Figs. 3 and 4.

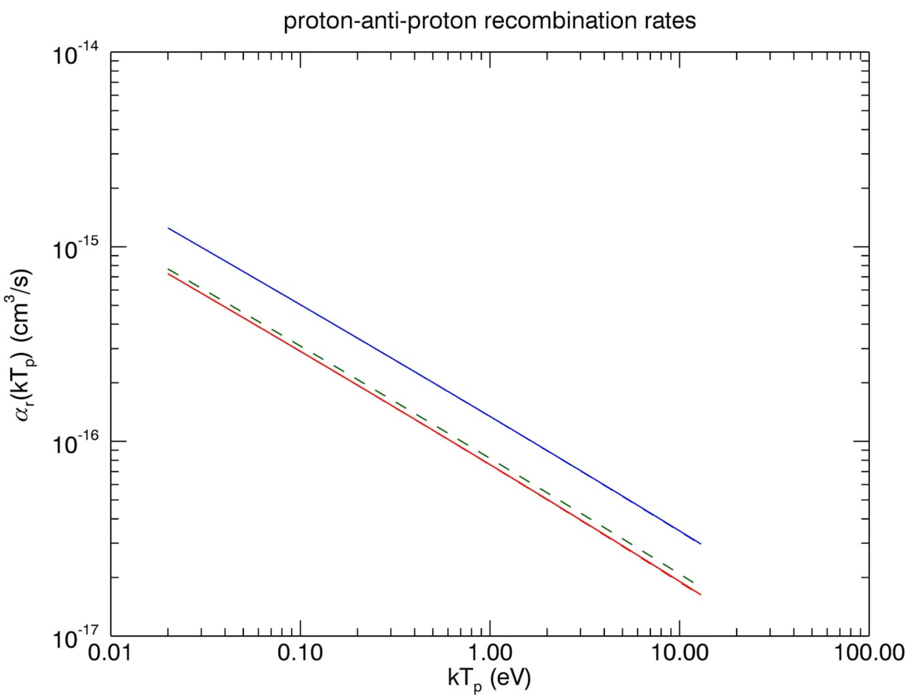

Figure 3: The comparison of recombination rates for $p\bar{p}$ as a function of proton temperature as in Fig. 2 (no dotted lines are shown).

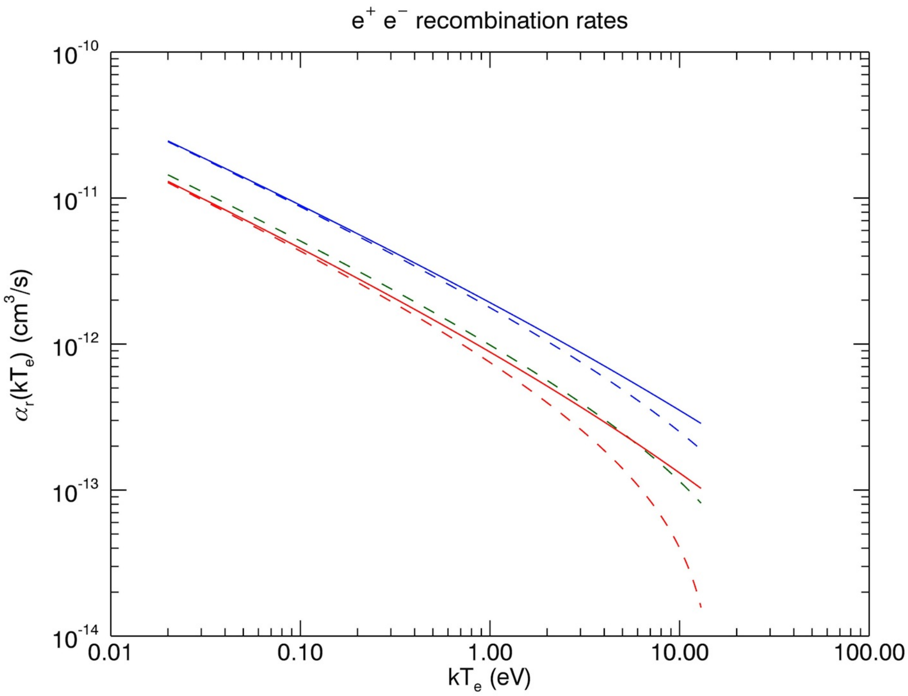

Figure 4: The comparison of recombination rates for $(e^+e^-)_{\mathrm{Ps}}$ as a function of electron temperature as in Fig. 2 (no dotted lines are shown).