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FINAL REPORT: Multigrid for Systems and Time-Dependent PDEs

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Abstract

This report has two sections. The first section is the motivation for looking at differing discretizations on coarse grids for solving a parabolic equation using multigrid in time. The second section contains selected numerical results from the many experiments conducted. The most interesting result is that for explicit fine grid discretizations, the best coarse discretization (i.e. smallest convergence rates) is a weighting between implicit and explicit methods.

1 Background: Coarse grid operators by RAP

Consider the parabolic PDE, $u_t - u_{xx} = f$ on a space-time mesh with $\Delta x = h$ and $\Delta t = \tau$. A common discretization is the explicit forward Euler method (FE)

$$\frac{u_i^k - u_i^{k-1}}{\tau} - \frac{u_{i-1}^{k-1} - 2u_i^{k-1} + u_{i+1}^{k-1}}{h^2} = f_i \tag{1}$$

Here subscripts correspond to spatial indexing and superscripts to temporal indexing. Multiplying by τ, we can write as

$$u_i^k - u^{k-1} - \alpha(u_{i-1}^{k-1} - 2u_i^{k-1} + u_{i+1}^{k-1}) = \tau f_i \tag{2}$$

where $\alpha = \tau/h^2$. The left-hand side can be written as a space-time stencil

$$FE = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & 2\alpha - 1 & -\alpha \end{bmatrix} \tag{3}$$

Other common discretizations can be written is stencil form in a similar way. The stencil for implicit backward Euler is

$$BE = \begin{bmatrix} 0 & 0 & 0 \\ -\alpha & 2\alpha+1 & -\alpha \\ 0 & -1 & 0 \end{bmatrix} \tag{4}$$

and the stencil for Crank-Nicholson ($CN = BE/2 + FE/2$) is

$$CN = \begin{bmatrix} 0 & 0 & 0 \\ -\alpha/2 & \alpha+1 & -\alpha/2 \\ -\alpha/2 & \alpha-1 & -\alpha/2 \end{bmatrix} \tag{5}$$

Assuming the grid is coarsened only in time by a factor of two, reasonable choices for the interpolation and restriction operators are based on the ideal operators from a cyclic reduction point of view with BOXMG style stencil collapsing. The ideal interpolation interpolates a fine point from the previous time step, the ideal restriction restricts a fine point residual to the next time step. All weights (interpolation and restriction) are 1.

After choosing a interpolation and restriction operator, the coarse grid operator defined by RAP may, or may not, correspond to rediscretizing the PDE using the same method as used on the fine grid. For example, if backward Euler, $A = BE$, is used on the fine grid and interpolation is based on the ideal interpolation, $P = P_i$, and restriction is based on the transpose $R = (P_i)^T$, then the coarse grid stencil will be

$$\begin{bmatrix} 0 & 0 & 0 \\ -2\alpha & 4\alpha+1 & -2\alpha \\ 0 & -1 & 0 \end{bmatrix}. \tag{6}$$

This is simply BE on the coarser mesh where $h_c = h, \tau_c = 2\tau$ so $\alpha_c = 2\alpha$. However, if forward Euler, $A = FE$, is used on the fine grid and interpolation is based on the ideal interpolation, $P = P_i$, and restriction is based on the

Table 1: Coarse grid operator RAP if forward Euler on fine grid

	$P = P_i$	$P = (R_i)^T$
$R = (P_i)^T$	CN	
$R = R_i$	FE	CN

Table 2: Coarse grid operator RAP if backward Euler on fine grid

	$P = P_i$	$P = (R_i)^T$
$R = (P_i)^T$	BE	
$R = R_i$	CN	BE

transpose $R = (P_i)^T$, then the coarse grid stencil will be

$$\begin{bmatrix} 0 & 0 & 0 \\ -\alpha & 2\alpha + 1 & -\alpha \\ -\alpha & 2\alpha - 1 & -\alpha \end{bmatrix}. \tag{7}$$

This is now CN on the coarse grid.

The situation when Crank-Nicholson is used in the fine grid is somewhat more complicated. As noted earlier, Crank-Nicholson is an equal averaging of the forward Euler and backward Euler methods, so $CN = BE/2 + FE/2$. Adding a weighting parameter, we define $CN(\rho) = (1 - \rho)BE + \rho FE$. All the previous methods can be written in this form ($BE = CN(0), FE = CN(1), CN = CN(1/2)$). With this definition, the relationship between the fine and coarse operators are summarized in the tables **??** and **??** and **??**. We see that weighting between the explicit and implicit methods may well change on the coarse grid.

Table 3: Coarse grid operator RAP if Crank-Nicholson on fine grid

	$P = P_i$	$P = (R_i)^T$
$R = (P_i)^T$	CN(1/4)	
$R = R_i$	CN(3/4)	CN(1/4)

2 Numerical experiments with differing discretizations

In this section we report the two level convergence of the methods described in the previous background section. Guided by the discussion, we experiment with differing choices of fine and coarse grid discretizations of the heat equation. In all experiments we use FCF relaxation followed by coarse grid correction and report the average convergence factor for the first 10 cycles. The domain for all problems was $0 \leq x \leq 1; 0 \leq t \leq 1$ with Dirichlet boundary condition in space.

2.1 Fine grid discretization by Forward Euler

The results for a fully explicit fine grid discretization depended strongly on the CFL parameter α. If the CFL condition $\alpha < 0.5$ held on both the coarse and fine grids, then convergence could be obtained using various coarse grid discretizations. For example, on a 50×20000 fine grid ($\alpha = .125$), we saw convergence factors of 0.1682 using BE on the coarse grid, 0.0762 using CN, and 0.1854 using FE. The results for various values of the weighting between implicit and explicit, ρ, on the coarse grid are shown in Figure 3.

Performing the same set of experiments on a 70×20000 fine grid ($\alpha = .245$), we get the results in Figure 4. Here the coarse grid $\alpha = .49$ is approaching the CFL condition for stability. We see much more variation in the observed convergence factors, with the best convergence attained quite close to CN ($\rho = 0.5$).

Performing the same set of experiments on a 99×20000 fine grid ($\alpha = .490$), we get the results in Figure 5. Here the coarse grid $\alpha = .9$ violates the

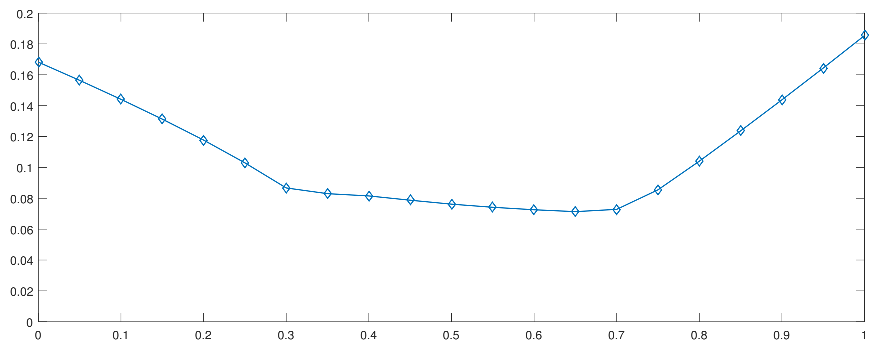

Figure 1: Observed convergence factors vs implicit weight ρ on coarse grid. Fine grid discretization is FE with $\alpha = 0.125$.

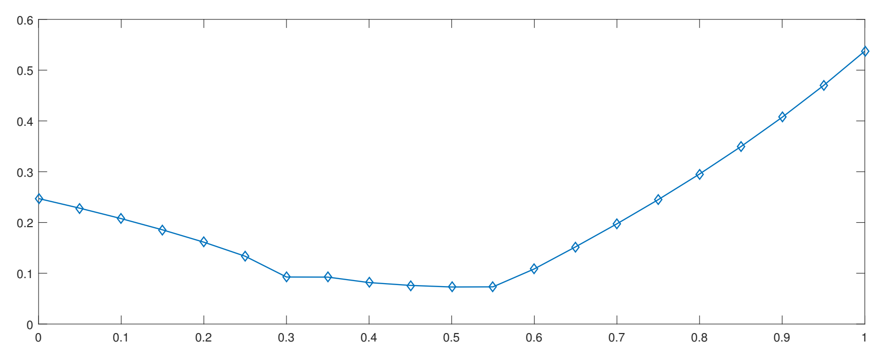

Figure 2: Observed convergence factors vs implicit weight ρ on coarse grid. Fine grid discretization is FE with $\alpha = 0.245$.

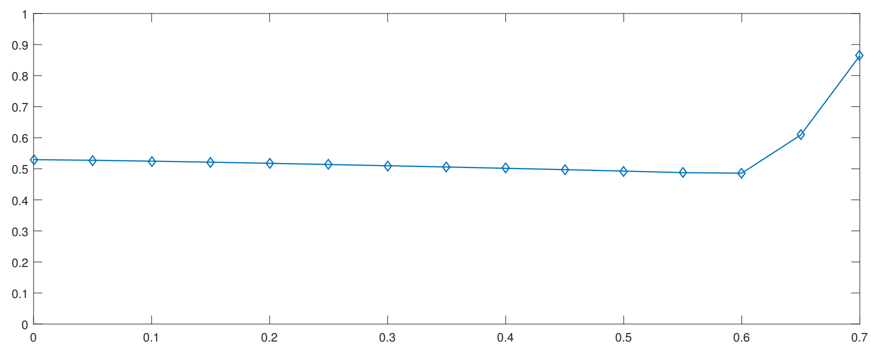

Figure 3: Observed convergence factors vs implicit weight ρ on coarse grid. Fine grid discretization is FE with $\alpha = 0.49$. The method diverged for $\rho > 0.7$.

CFL condition for stability. We see little change in the convergence factor until we observe divergence for $\rho > 0.7$.

2.2 Fine grid discretization by Backward Euler

If a fully implicit method is used on the fine grid we saw generally better performance using the same method on the coarse grid. For example, on a 50×20000 grid, we saw convergence factors of 0.0723 using BE on the coarse grid, 0.1283 using CN, and 0.3111 using FE. Note that for this grid FE is stable on the coarse grid. It was possible to see mildly better convergence factors by adjusting the weighting between implicit and explicit, ρ. For this example, $\rho = .25$ gave a convergence factor of 0.0685. The fully implicit methods allow bigger time steps and faster coarsening, so it seems likely using them on the coarse grid is the right choice here. When the CFL condition $\alpha < 0.5$ is violated on the coarse grid, the FE and CN options may well diverge.